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Some New Proof System for Propositional Modal Logic

(Submitted by academician Yu. H. Shoukourian 5/II 2024)

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1. Introduction. Some uniform proof systems E on the base of determinative disjunctive normal forms and some elimination rules have been formerly constructed for propositional two-valued Classical, Intuitionistic, Minimal Logics [1, 2] and some Many-valued Logics [3]. These systems are dual to the resolution systems, but the preference of the mentioned systems is in the possibility of easily receiving the lower exponential bounds for proof complexities of many tautology classes. For the construction of systems type E in any logic, it is necessary to define:

- the concept of literals, through which the determinative conjuncts will be formed,
- the opposite literals, through which the inference rules will be defined.

In this paper a system of type E is defined for propositional Modal Logic. If previously Modal Logic was mainly used for linguistic structures, particularly, their truth, possibility, necessity, temporal judgments, as well as for the study of moral and ethical issues, then recently it has begun to be actively applied in various fields of computer science. Particularly, for the selection of program execution directions, formalization or for representing the dynamic properties of transitioning from one situation to another.

2. Preliminaries. Many systems for representing the propositional modal logic are known. Except the logical connectives of unimodal logic these systems use two additional logical connectives: $\Diamond$ - 'is possible' and $\Box$ - 'is necessary', on the base of which are introduced $\Rightarrow$ - ‘strong implication’ and $\Leftrightarrow$ - ‘strong equivalence’ as well.

Our result is based on the most popular system S4. First, let's give some definitions.

The logical connectives in S4 are $\neg$, $\wedge$, and $\Diamond$.

The definition of a formula is as follows:

- any logical variable is a formula,
- if P and Q are formulas, then $(\neg P)$, $(\Diamond P)$ and $(P \wedge Q)$ are formulas,
- there are no other formulas.

Other logical connectives are defined in the following ways:

1. $P \vee Q$ is defined as $\neg(\neg P \wedge \neg Q)$.
2. $P \rightarrow Q$ is defined as $\neg(P \wedge \neg Q)$.
3. $P \leftrightarrow Q$ is defined as $(P \rightarrow Q) \wedge (Q \rightarrow P)$.
4. $P \Rightarrow Q$ is defined as $\neg \Diamond (P \wedge \neg Q)$.
5. $P \Leftrightarrow Q$ is defined as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
6. $\Box P$ is defined as $\neg \Diamond \neg P$.

The following function is also introduced for the future use:

$(\Diamond P)^{\sigma}$ as $(\Diamond P) \Leftrightarrow \sigma$. It is not difficult to see that $(\Diamond P)^{1}\ is\ (\Diamond P)$ and $(\Diamond P)^{0}$ is $\neg(\Diamond P)$.

The axioms of S4 are:

1. $p \wedge q \Rightarrow p,$
2. $p \wedge q \Rightarrow q \wedge p,$
3. $[(p \wedge q) \wedge r] \Rightarrow [p \wedge (q \wedge r)],$
4. $p \Rightarrow (p \wedge p),$
5. $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r),$
6. $\Diamond\Diamond\, p \Rightarrow \Diamond\, p.$

The inference rules are:

1. *Substitution.* A derived formula remains derivable if any logical variable in it is replaced everywhere in the formula.
2. *Union.*
$$\frac{P,\ Q}{P \wedge Q}$$
3. Separation.
$$\frac{P,\ P \Rightarrow Q}{Q}$$
4. *Replacement.* If $P \Leftrightarrow Q$ is derived, then the derived formula remains derivable if some entries of Q in it are replaced by the formula P.

Let us describe the method of solvability of the S4 system developed by Anderson [4].

1. First, the formula is brought to the normal form, in which

- there are no logical connectives other than $\neg$, $\wedge$ and $\Diamond$,
- no subformula has the form of $\neg\neg\alpha$ or $\Diamond \neg(\alpha \wedge \beta)$.

It is not difficult to see that it can be done on the base of some formulas derived in the system S4 (see [4,5]).

2. To all **constituents-logical variables and subformulas of the form $\Diamond\, \alpha$ for any variable α** – we assign arbitrary values from the set {0,1} in all possible ways.
3. The following three types of **invalids** are removed from the sets of assigned values:

- since the formula $p \to\Diamond\, p$ can be proved in the system S4, then the set in which p is assigned to 1 and $\Diamond\, p$ is assigned to 0 is invalid,
- since in the system S4 the formula $\Diamond\, P \to (\Diamond\, Q_1 \vee \ldots \vee\Diamond\, Q_m)$ can be derived from the formula $P \to (Q_1 \vee \ldots \vee Q_m)$ and the formula $\Diamond\, P \to\Diamond\, Q$ can be derived from the formula $P \to\Diamond\, Q$, then the set in which every $\Diamond\, Q_i$ is assigned to 0 and $\Diamond\, P$ is assigned to 1 is invalid,
- since in the system S4 the formula $\Box P$ can be derived from the formula P and the formula $\neg\Diamond\, P$ can be derived from the formula $\neg P$, then the set in which $\neg P$ is assigned to 1 and $\Diamond\, P$ is assigned to 1 is invalid.

It is known that a formula can be proved in a system S4 (is a modal tautology) if and only if, for all valid sets of values assigned to all its constituents, the formula takes the value 1.

3. Main Results. Now we can define the concept of determinative conjunct and determinative disjunctive normal form for modal logic.

We call a *replacement-identities* each of the following trivial identities for a propositional modal formula g

$0 \wedge g = 0,\quad g \wedge 0 = 0,\quad 1 \wedge g = g,\quad g \wedge 1 = g,\quad g \wedge g = g,$
$\neg 0 = 1,\quad \neg 1 = 0,\quad \neg\neg g = g,\quad \Diamond\, 1 = 1,\quad \Diamond\Diamond\, g =\Diamond\, g,$

and replacement relation $\Diamond\, 0 \leq 1$, which can be presented as two variants of identities: $\Diamond\, 0 =1$ or $\Diamond\, 0 = 0$.

Application of a replacement-identities to some word consists in replacing some its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

Let φ be a formula of a modal propositional logic, $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the **set of all constituents** of that formula and $A' = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}\}$ (where $1 \leq m \leq n$) be the subset of A.

Definition 2.3. If $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \in E^m$ is valid set of values (see 3.1) for constituents $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}$ of formula φ, then modal conjunct $K^\sigma = \{\alpha_{i_1}^{\sigma_1}, \alpha_{i_2}^{\sigma_2}, \ldots, \alpha_{i_m}^{\sigma_m}\}$ will be called a $\varphi -1$-determinative ($\varphi-0$-determinative) if assigning σ_j $(1 \leq j \leq m)$ to each α_{i_j} and successively using replacement identities (one or both variants of replacement relation) we obtain the value of φ (1 or 0) independently of the values of the remaining constituents.

$\varphi-1$-determinative conjunct and $\varphi-0$-determinative conjunct are also called φ-determinative or determinative for φ.

DNF $D = \{K_1, K_2, \ldots, K_l\}$ is called a determinative disjunctive normal form (dDNF) for φ if φ and D are semantically equivalent and every conjunct K_i $(1 \leq i \leq r)$ is 1-determinative for φ.

Definition of the system E_{mod}:

The axioms of E_{mod} are not fixed, but for any formula φ, each conjunct of its some dDNF can be considered as an axiom.

Elimination rule (ε-rule) represents the following rules:

$$\frac{K_1 \cup p \quad K_2 \cup \neg p}{K_1 \cup K_2},$$

$$\frac{K_1 \cup \Diamond p \quad K_2 \cup \neg \Diamond p}{K_1 \cup K_2},$$

$$\frac{K_1 \cup p \quad K_2 \cup \neg \Diamond p}{K_1 \cup K_2},$$

$$\frac{K_1 \cup \neg p \quad K_2 \cup \Diamond p}{K_1 \cup K_2},$$

where K_1 and K_1 are conjuncts and p is a variable.

The proof in E_{mod} is a finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of E_{mod}, or is inferred from earlier conjuncts in the sequence by one of the ε-rules.

DNF $D = \{K_1, K_2, \dots, K_l\}$ is called tautology if using the ε-rules, one can prove the empty conjunct ($\emptyset$) from the axioms $\{K_1, K_2, \dots, K_l\}$. It is not difficult to prove that system E_{mod} is full and sound.

Taking into consideration that every classical tautology is modal tautology and using the results of [1] it is not difficult to prove the following.

Theorem. *For sufficiently large* n *there are sequences of tautologies such, that*

1) size of them is n by order,

2) *number of lines in any E_{mod} -proofs of them is at least 2^n by order.*

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Some New Proof System for Propositional Modal Logic

A method of constructing some proof system for propositional modal logic is described. Earlier for unmodal logics introduced notions of determinative conjunct and determinative disjunctive normal, as well as the elimination rule, are generalized for propositional modal logic and on the base of them, the proof system E_{mod} is constructed. For some sequences of tautologies, lower exponential bounds for the number of proof lines in the described system are easily obtained.

Հ. Ա. Թամազյան

Ասույթային մոդալ տրամաբանության արտածումների նոր համակարգ

Նկարագրված է արտածման որոշակի համակարգի կառուցման եղանակը ասույթային մոդալ տրամաբանության համար: Ընդհանրացված են նախկինում ոչ մոդալ տրամաբանությունների համար ներմուծված որոշիչ կոնյունկտի և որոշիչ դիզյունկտիվ նորմալ ձևի, ինչպես նաև բացառման կանոնը, գաղափարները, և դրանց հիման վրա կառուցված է E_{mod} արտածման համակարգը: Նույնաբանությունների որոշակի հաջորդականությունների համար հեշտ ստացված է ցուցչային կարգի ստորին գնահատական նկարագրված համակարգում արտածման նվազագույն քայլերի քանակի համար:

А. А. Тамазян

Новая система выводов для пропозициональной модальной логики

Описан метод построения некоторой пропозициональной системы выводов для модальной логики. Введенные ранее для немодальных логик понятия определяющего конъюнкта и определяющей дизъюнктивной нормальной формы, а также правила элиминации обобщены для пропозициональной модальной логики, и на их основе построена система E_{mod}. Для некоторых последовательностей тавтологий легко получены нижние экспоненциальные оценки минимального количества шагов выводов в описываемой системе.

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