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Duality of Energy and Probability in Finite-Volume Models of Statistical Physics

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Introduction. It is well-known that the Gibbs formula (which establishes a relationship between probability and energy) is the basis of statistical physics. Much attention has been paid to the justification of the Gibbs formula using physical reasoning. In [1], it was shown that the Gibbs formula can have a purely mathematical justification for both finite and infinite systems (for the case of finite-volume systems, see also [2]). In our paper, we will show that there is a deeper relationship between energy and probability, namely, energy and probability are dual concepts.

Duality in mathematics is the principle according to which any true statement of one theory corresponds to a true statement in the dual theory. Here, we will show how this principle can be applied to solve the known problem of describing a finite random field by a set of consistent conditional distributions (see, for example, [3]). A direct probabilistic solution to this problem is given in [2].

1. Duality of energy and probability in finite volume. Let Λ be a set with a finite number of elements, $1 < |\Lambda| < \infty$, and let each point $t \in \Lambda$ be associated with the set X^t, which is a copy of some finite set X. Denote by $X^\Lambda = \{x = (x_t, t \in \Lambda) : x_t \in X, t \in \Lambda\}$ the set of functions (configurations) defined on Λ and tacking values in X. For any $V \subset \Lambda$, denote by x_V the restriction of configuration $x \in X^\Lambda$ on V. For any $V, I \subset \Lambda$ such that $V \cap I = \emptyset$, and any $x \in X^V$, $y \in X^I$, denote by xy the concatenation of x with y, that is, the configuration on $V \cup I$ equal to x on V and to y on I. For one-point sets $\{t\}$, $t \in \Lambda$, braces will be omitted.

Probability distribution on X^Λ is a function $P_\Lambda : X^\Lambda \to [0,1]$ satisfying the following conditions:

$$P_\Lambda(x) > 0, x \in X^\Lambda, \qquad \sum_{x\in X^\Lambda} P_\Lambda(x) = 1. \tag{1}$$

Probability distribution P_Λ on X^Λ sometimes will be called a (*finite*) *random field*.

A function $\Delta_\Lambda: X^\Lambda \times X^\Lambda \to \mathbb{R}$ satisfying

$$\Delta_\Lambda(x,u) = \Delta_\Lambda(x,z) + \Delta_\Lambda(z,u), \qquad x,u,z \in X^\Lambda, \tag{2}$$

will be called a *transition energy*. The value $\Delta_\Lambda(x,u)$ of this function can be interpreted as an amount of energy needed to change the state of the physical system from x to u (in the finite volume Λ).

The following result establishes a relationship between two fundamental concepts: energy and probability.

Theorem 1. *For a set $P_\Lambda = \{P_\Lambda(x), x \in X^\Lambda\}$ of numbers to be a probability distribution on X^Λ it is necessary and sufficient that elements of P_Λ have the Gibbs form*

$$P_\Lambda(x) = \frac{exp\{\Delta_\Lambda(x,u)\}}{\sum_{z\in X^\Lambda} exp\{\Delta_\Lambda(z,u)\}}, \qquad x \in X^\Lambda, \tag{3}$$

where $u \in X^\Lambda$ and $\Delta_\Lambda = \{\Delta_\Lambda(x,u), x,u \in X^\Lambda\}$ is a transition energy on $X^\Lambda \times X^\Lambda$ with

$$\Delta_\Lambda(x,u) = \ln\frac{P_\Lambda(x)}{P_\Lambda(u)}, \qquad x,u \in X^\Lambda.$$

Since Δ_Λ satisfies (2), there is a function $H_\Lambda = \{H_\Lambda(x), x \in X^\Lambda\}$ such that

$$\Delta_\Lambda(x,u) = H_\Lambda(u) - H_\Lambda(x), \qquad x \in X^\Lambda. \tag{4}$$

Substituting (4) into (3), we obtain

$$P_\Lambda(x) = \frac{exp\{-H_\Lambda(x)\}}{\sum_{z\in X^\Lambda} exp\{-H_\Lambda(z)\}}, \qquad x \in X^\Lambda,$$

where H_Λ can be considered as a Hamiltonian (potential energy) of a physical system. Hence, in the case of finite volume Λ, any function H_Λ on X^Λ can be interpreted as a Hamiltonian (see [1]). Particularly, in the classical interpretation,

$$H_\Lambda(x) = \sum_{t,s\in\Lambda} \Phi_{\{t,s\}}(x_t x_s), \qquad x \in X^\Lambda,$$

where Φ is a pair interaction potential.

The relationship between probability distribution and transition energy can be formulated in terms of operators. Let $\mathcal{P} = \{P_\Lambda\}$ be the set of all probability distributions on X^Λ and let $\mathcal{D} = \{\Delta_\Lambda\}$ be the set of all transition energies on $X^\Lambda \times X^\Lambda$. Consider the operator $T: \mathcal{P} \to \mathcal{D}$ which maps an element from $\mathcal{P}$ to an element from $\mathcal{D}$ according to the formula

$$(TP_\Lambda)(x,u) = \ln \frac{P_\Lambda(x)}{P_\Lambda(u)}, \qquad x,u \in X^\Lambda,$$

and the operator $T^{-1}: \mathcal{D} \to \mathcal{P}$ which maps an element from $\mathcal{D}$ to an element from $\mathcal{P}$ by the formula

$$(T^{-1}\Delta_\Lambda)(x) = \frac{exp\{\Delta_\Lambda(x,u)\}}{\sum_{z \in X^\Lambda} exp\{\Delta_\Lambda(z,u)\}}, \qquad x \in X^\Lambda,$$

where $u \in X^\Lambda$. Due to condition (2), the operator T^{-1} is correctly defined. It is clear that both operators T and T^{-1} depend on Λ, but to simplify the notations, sometimes we will not directly specify this dependence.

The following statement holds true.

Proposition. *Operators T and T^{-1} are mutually inverse, that is, for all $P_\Lambda \in \mathcal{P}$ and $\Delta_\Lambda \in \mathcal{D}$, it holds*

$$T^{-1}TP_\Lambda = P_\Lambda, \qquad TT^{-1}\Delta_\Lambda = \Delta_\Lambda.$$

It is easy to see that for any $P_\Lambda \in \mathcal{P}$, function TP_Λ satisfies the characteristic property (2) of transition energies, while for any $\Delta_\Lambda \in \mathcal{D}$, function $T^{-1}\Delta_\Lambda$ satisfies (1), which characterizes a probability distribution. Therefore, any statement about probability P_Λ can be formulated in terms of corresponding transition energy Δ_Λ, and vise versa.

2. Duality of transition energy field and conditional distribution. Let P_Λ be a probability distribution on X^Λ. There is a set $Q(P_\Lambda) = \left\{Q_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subset \Lambda\right\}$ of its conditional probabilities

$$Q_V^{\bar{x}}(x) = \frac{P_\Lambda(x\bar{x})}{\sum_{z \in X^V} P_\Lambda(z\bar{x})}, \qquad x \in X^V, \bar{x} \in X^{\Lambda \setminus V}, V \subset \Lambda.$$

It is clear, that for any fixed $V \subset \Lambda$ and $\bar{x} \in X^{\Lambda \setminus V}$, function $Q_V^{\bar{x}}$ is a probability distribution on X^V. We will also consider the set $Q_1(P_\Lambda) = \left\{Q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\right\} \subset Q(P_\Lambda)$ of one-point conditional probabilities generated by P_Λ.

Now, let $Q = \left\{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subset \Lambda\right\}$ be a set of probability distributions $q_V^{\bar{x}}$ on X^V parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus V}$, $V \subset \Lambda$. A natural question arises: does there exist a probability distribution P_Λ on X^Λ for which Q is a set of its conditional probabilities, that is, $Q(P_\Lambda) = Q$? The answer is given by the following statement.

Theorem 2. *Let $Q = \left\{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subset \Lambda\right\}$ be a set of probability distributions on X^V parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus V}$, $V \subset \Lambda$. There exists a unique probability distribution P_Λ on X^Λ such that $Q(P_\Lambda) = Q$ if and only if the elements of Q satisfy the following consistency conditions: for* *any disjoint $V, I \subset \Lambda$ and $\bar{x} \in X^{\Lambda\setminus(V\cup I)}$, $x, u \in X^V$, $y \in X^I$, it holds*

$$q_{V\cup I}^{\bar{x}}(xy)q_V^{\bar{x}y}(u) = q_{V\cup I}^{\bar{x}}(uy)q_V^{\bar{x}y}(x). \tag{5}$$

Condition (5) is a finite-volume version of the well-known R. Dobrushin's consistency condition, see [4]. The set $Q = \{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda\}$ of probability distributions satisfying (5) is called a *finite-volume specification*. Theorem 2 states that any finite-volume specification is a set of conditional probabilities of some (uniquely determined) joint distribution.

Let Δ_Λ be a transition energy on $X^\Lambda \times X^\Lambda$. Consider the set $D(\Delta_\Lambda) = \{\Delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda\}$ of functions

$$\Delta_V^{\bar{x}}(x, u) = \Delta_\Lambda(x\bar{x}, u\bar{x}), \qquad x, u \in X^V, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda.$$

It is not difficult to see that for any fixed $V \subset \Lambda$ and $\bar{x} \in X^{\Lambda\setminus V}$, function $\Delta_V^{\bar{x}}$ is a transition energy on $X^V \times X^V$, that is,

$$\Delta_V^{\bar{x}}(x, u) = \Delta_V^{\bar{x}}(x, z) + \Delta_V^{\bar{x}}(z, u), \qquad x, u, z \in X^V. \tag{6}$$

Now, let us consider a set $D = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda\}$ of transition energies $\delta_V^{\bar{x}}$ on $X^V \times X^V$ parameterized by boundary conditions $\bar{x} \in X^{\Lambda\setminus V}$, $V \subset \Lambda$. The following statement holds true (see also [1]).

Theorem 3. *Let $D = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda\}$ be a set of transition energies $\delta_V^{\bar{x}}$ on $X^V \times X^V$ parameterized by boundary conditions $\bar{x} \in X^{\Lambda\setminus V}$, $V \subset \Lambda$. There exists a unique transition energy Δ_Λ on $X^\Lambda \times X^\Lambda$ such that $D(\Delta_\Lambda) = D$ if and only if the elements of D satisfy the following consistency conditions: for any disjoint $V, I \subset \Lambda$ and $\bar{x} \in X^{\Lambda\setminus(V\cup I)}$, $x, u \in X^V$, $y \in X^I$, it holds*

$$\delta_{V\cup I}^{\bar{x}}(xy, uy) = \delta_V^{\bar{x}y}(x, u). \tag{7}$$

The set $D = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda\}$ of transition energies satisfying (7) is called a *finite-volume transition energy field*. This notion was introduced in [1] for the case of systems defined in infinite volume (on the integer lattice $\mathbb{Z}^d$, $d \geq 1$).

Previously established duality of probability P_Λ and energy Δ_Λ allows establishing the one-to-one correspondence between systems Q and D. Namely, for every fixed $V \subset \Lambda$, define operators $T_V: \{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}\} \to \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}\}$ and $T_V^{-1}: \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}\} \to \{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}\}$ by

$$(T_V q_V^{\bar{x}})(x, u) = \ln\frac{q_V^{\bar{x}}(x)}{q_V^{\bar{x}}(u)}, \quad (T_V^{-1}\delta_V^{\bar{x}})(x) = \frac{exp\{\delta_V^{\bar{x}}(x,u)\}}{\sum_{z\in X^V} exp\{\delta_V^{\bar{x}}(x,u)\}}, \quad x, u \in X^\Lambda. \tag{8}$$

Then operators $T: Q \to D$ and $T^{-1}: D \to Q$ defined by

$$Tq_V^{\bar{x}} = T_V q_V^{\bar{x}}, \qquad T^{-1}\delta_V^{\bar{x}} = T_V^{-1}\delta_V^{\bar{x}}, \qquad \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda,$$

are mutually inverse. Moreover, the elements of Q satisfy conditions (5) if and only if the elements of D satisfy conditions (7). That means that there is a duality between specification (conditional distribution) and transition energy field.

Further, we will establish one of the important properties of the transition energy – its additivity. Let $D = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \subset \Lambda\}$ be a transition energy field. Then for any disjoint $V, I \subset \Lambda$ and $\bar{x} \in X^{\Lambda\setminus(V\cup I)}$, $x, u \in X^V$, $y, v \in X^I$, using (6) and (7), we can write

$$\delta_{V\cup I}^{\bar{x}}(xy, uv) = \delta_{V\cup I}^{\bar{x}}(xy, uy) + \delta_{V\cup I}^{\bar{x}}(uy, uv) = \delta_V^{\bar{x}y}(x, u) + \delta_I^{\bar{x}u}(y, v)$$

and

$$\delta_{V\cup I}^{\bar{x}}(xy, uv) = \delta_{V\cup I}^{\bar{x}}(xy, xv) + \delta_{V\cup I}^{\bar{x}}(xv, uv) = \delta_I^{\bar{x}x}(y, v) + \delta_V^{\bar{x}v}(x, u).$$

From here it follows, that for the elements of the one-point subsystem $\{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\} \subset D$, one has

$$\delta_t^{\bar{x}y}(x, u) + \delta_s^{\bar{x}u}(y, v) = \delta_s^{\bar{x}x}(y, v) + \delta_t^{\bar{x}v}(x, u) \tag{9}$$

for any $x, u \in X^t$, $y, v \in X^s$, $\bar{x} \in X^{\Lambda\setminus\{t,s\}}$, $t, s \in \Lambda$. Relation (9) has a simple physical meaning. There are two ways to change the state of the system in $\{t, s\}$ from xy to uv with the state $\bar{x}$ in $\Lambda\setminus\{t, s\}$ unchanged. First, change the state of the system at point t from x to u under boundary condition $y\bar{x}$, and then at point s from y to v already under boundary condition $u\bar{x}$. Or, starting from point s, change the state from y to v under the boundary condition $x\bar{x}$, and then, under the boundary condition $v\bar{x}$, change it at point t from x to u. Naturally, the same amount of energy must be spent in both cases.

A set $D_1 = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ of one-point transition energies $\delta_t^{\bar{x}}$ on $X^t \times X^t$ parameterized by boundary conditions $\bar{x} \in X^{\Lambda\setminus t}$, $t \in \Lambda$, and satisfying consistency conditions (9) is called a (*finite-volume*) *one-point transition energy field* (see also [1, 2]).

Theorem 4. *A function Δ_Λ on $X^\Lambda \times X^\Lambda$ is a transition energy if and only if it can be represented in the form*

$$\Delta_\Lambda(x, u) = \delta_{t_1}^{x_{\Lambda\setminus t_1}}(x_t, u_{t_1}) + \delta_{t_2}^{u_{t_1}x_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2}, u_{t_2}) + \cdots + \delta_{t_n}^{u_{\Lambda\setminus t_n}}(x_{t_n}, u_{t_n}),$$

where $\Lambda = \{t_1, t_2, \dots, t_n\}$ is some enumeration of points in Λ, $|\Lambda| = n$, and $D_1 = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ is a one-point transition energy field.

3. Application of the duality. In this section, we will show how the established duality between the transition energy and probability distribution can be applied to solve a known problem of the description of a finite random field by a set of consistent (one-point) conditional distributions.

This problem was considered by many authors. In the well-known paper [3] by S. Geman and D. Geman, it was divided into two questions (tasks). First, how one can define (compute) a joint distribution knowing its conditionals? And second, the most difficult one, how one can spoil conditional distributions, that is, when a given set of functions are conditional probabilities for some (necessary unique) distribution on X^{Λ}?

As it was mentioned above, the characteristic property (5) of conditional probabilities was known and successfully applied to the problem of describing lattice random fields by specifications (see [4]). However, one cannot derive the characteristic property of one-point conditional probabilities from (5), and such property remained unknown for a long time. The consistency conditions for a set of one-point probability distributions parameterized by boundary conditions to be a one-point subset of some (uniquely determined) specification were introduced in [5] for the case of infinite systems.

The solution to the problem of the describing finite random field by a set of consistent one-point conditional distributions was given in [2] using a purely probabilistic approach. Below, we will give the solution to this problem based on the duality between transition energy and probability.

Let $Q_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ be a set of probability distributions $q_t^{\bar{x}}$ on X^t parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus t}$, $t \in \Lambda$, and let $D_1 = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ be a one-point transition energy field. Consider the mutually inverse operators $T_1 = \{T_t, t \in \Lambda\}: Q_1 \to D_1$ and $T_1^{-1} = \{T_t^{-1}, t \in \Lambda\}: D_1 \to Q_1$ where T_t and T_t^{-1}, $t \in \Lambda$, are defined by (8):

$$\delta_t^{\bar{x}}(x,u) = \left(T_1 q_t^{\bar{x}}\right)(x,u), \quad q_t^{\bar{x}}(x) = \left(T_1^{-1}\delta_t^{\bar{x}}\right)(x), \quad x,u \in X^t, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda.$$

Since the elements of D_1 satisfy condition (7), the elements of Q_1 cannot be arbitrary and have to satisfy appropriate consistency conditions. To find such conditions, we note that for all $t,s \in \Lambda$, $\bar{x} \in X^{\Lambda \setminus \{t,s\}}$ and $x,u \in X^t$, $y,v \in X^s$, one has

$$\delta_t^{\bar{x}y}(x,u) + \delta_s^{\bar{x}u}(y,v) = \left(T_1 q_t^{\bar{x}y}\right)(x,u) + \left(T_1 q_s^{\bar{x}u}\right)(y,v)$$
$$= \ln\left(\frac{q_t^{\bar{x}y}(x)}{q_t^{\bar{x}y}(u)} \cdot \frac{q_s^{\bar{x}u}(y)}{q_s^{\bar{x}u}(v)}\right)$$

and

$$\delta_s^{\bar{x}x}(y,v) + \delta_t^{\bar{x}v}(x,u) = \left(T_1 q_s^{\bar{x}x}\right)(y,v) + \left(T_1 q_t^{\bar{x}v}\right)(x,u)$$
$$= \ln\left(\frac{q_s^{\bar{x}x}(y)}{q_s^{\bar{x}x}(v)} \cdot \frac{q_t^{\bar{x}v}(x)}{q_t^{\bar{x}v}(u)}\right).$$

Hence, the elements of D_1 satisfy condition (7) if and only if the elements of Q_1 satisfy the following consistency condition: for all $t,s \in \Lambda$, $\bar{x} \in X^{\Lambda \setminus \{t,s\}}$ and $x,u \in X^t$, $y,v \in X^s$ it holds

$$q_t^{\bar{x}y}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(u) q_s^{\bar{x}u}(y) = q_t^{\bar{x}y}(u) q_s^{\bar{x}u}(v) q_t^{\bar{x}v}(x) q_s^{\bar{x}x}(y). \quad (10)$$

A set $Q_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ of probability distributions $q_t^{\bar{x}}$ on X^t parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus t}$, $t \in \Lambda$, and satisfying the consistency conditions (10) is called a (*finite-volume*) *1-specification*.

Let us now show how the established relation between 1-specification and one-point transition energy field allows constructing the distribution P_Λ on X^Λ compatible with Q_1, that is, such P_Λ that $Q_1(P_\Lambda) = Q_1$.

First, let us find the connection between a probability distribution P_Λ and its one-point conditional probabilities $Q_1(P_\Lambda) = \{Q_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$. According to Theorems 1, there exists a unique transition energy Δ_Λ for P_Λ such that

$$P_\Lambda(x) = \frac{exp\{\Delta_\Lambda(x,u)\}}{\sum_{z\in X^\Lambda} exp\{\Delta_\Lambda(z,u)\}} = \left(\sum_{z\in X^\Lambda} exp\{\Delta_\Lambda(z,x)\}\right)^{-1},$$

where we used property (2) of Δ_Λ. Further, due to Theorem 4, there exists a one-point transition energy field $D_1(\Delta_\Lambda) = \{\Delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ such that

$$\sum_{z\in X^\Lambda} exp\{\Delta_\Lambda(z,x)\}$$

$$= \sum_{z\in X^\Lambda} exp\{\Delta_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1}, x_{t_1}) + \Delta_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2}, x_{t_2})$$

$$+ \cdots (z_{t_n}, x_{t_n})\}.$$

Note that by definitions of $D_1(\Delta_\Lambda)$ and $Q_1(P_\Lambda)$, we have

$$\Delta_t^{\bar{x}}(z,x) = \Delta_\Lambda(z\bar{x}, x\bar{x}) = \ln\frac{\mathrm{P}_\Lambda(z\bar{x})}{\mathrm{P}_\Lambda(x\bar{x})} = \ln\frac{Q_t^{\bar{x}}(z)}{Q_t^{\bar{x}}(x)}, \qquad x,z \in X^t, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda,$$

and hence,

$$P_\Lambda(x) = \left(\sum_{z\in X^\Lambda} \frac{Q_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1})}{Q_{t_1}^{z_{\Lambda\setminus t_1}}(x_{t_1})} \cdot \frac{Q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2})}{Q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2})} \cdot \ldots \cdot \frac{Q_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n})}{Q_{t_n}^{x_{\Lambda\setminus t_n}}(x_{t_n})}\right)^{-1}.$$

The obtained connection between P_Λ and $Q_1(P_\Lambda)$ can be used to define a probability distribution compatible with a given 1-specification. Namely, let $Q_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ be a 1-specification. For any $x \in X^\Lambda$, put

$$P_\Lambda(x) = \left(\sum_{z\in X^\Lambda} \frac{q_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1})}{q_{t_1}^{z_{\Lambda\setminus t_1}}(x_{t_1})} \cdot \frac{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2})}{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2})} \cdot \ldots \cdot \frac{q_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n})}{q_{t_n}^{x_{\Lambda\setminus t_n}}(x_{t_n})}\right)^{-1},$$

where $\Lambda = \{t_1, t_2, \ldots, t_n\}$ is some enumeration of the points of Λ, $n = |\Lambda|$. Due to (10), this formula is correct, that is, the values of P_Λ does not depend on the way of enumeration of the points in Λ. It is not difficult to see that P_Λ is a probability distribution on X^Λ. Finally, by direct computations, one can show that $Q_1(P_\Lambda) = Q_1$.

Hence, the additivity property of the transition energy allowed us to find the connection between the joint and conditional distributions, and the consistency conditions of the elements of the one-point transition energy field prompted the form of the consistency conditions of the elements of the one-point conditional distribution.

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Duality of Energy and Probability in Finite-Volume Models of Statistical Physics

It is shown that in the framework of mathematical physics, energy and probability are dual concepts. On this basis, a solution to the well-known problem of describing a finite random field by a set of consistent conditional distributions is given.

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Էներգիայի և հավանականության երկակիությունը վիճակագրական ֆիզիկայի վերջավոր մոդելներում

Ցույց է տրված, որ վիճակագրական ֆիզիկայի շրջանակում էներգիան և հավանականությունը երկակի հասկացություններ են: Օգտագործելով այս արդյունքը, լուծում է տրվում պայմանական բաշխումների համակարգի միջոցով վերջավոր պատահական դաշտի նկարագրման հայտնի խնդրին:

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Двойственность энергии и вероятности в конечных моделях статистической физики

Показано, что в рамках статистической физики энергия и вероятность – двойственные понятия. На этой основе приводится решение известной проблемы описания конечного случайного поля совокупностью согласованных условных распределений.

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