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# On Almost-Everywhere Convergence of Malmquist –Takenaka Series

(Submitted by academician G. G. Gevorkyan 28/II 2022)

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**Introduction.** Let $F$ be a holom function in a neighborhood of the closed unit disc. We consider the holomorphic function $F_1$ defined on the same neighborhood by

$$F(z) = F(0) + z F_1(z).$$

Iterating this process, one obtains the Fourier series on the unit circle. We can consider a more general iterative process where we replace the factor $z$ by a different Mobius transform of the unit disc vanishing at point $a_n$,

$$F_n(z) = F_n(a_n) + \frac{z-a_n}{1-\overline{a}_n z} F_{n+1}(z).$$

Consider a sequence of points $(a_n)_{n=1}^{\infty}$ inside the unit disk such that

$$\sum_{n=1}^{\infty} 1 - |a_n| = +\infty.$$

The associated Blaschke products and Malmquist – Takenaka (MT) basis are defined as $B_n(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a}_j z}$, $\varphi_n(z) = B_n(z) \frac{\sqrt{1-|a_{n+1}|^2}}{1-\overline{a}_{n+1} z}$.

Unlike the case of Fourier series, $B_n$'s are not orthonormal, but $\varphi_n$'s are due to the normalization (see Lemma 3 in [1]).

It was shown by Coifman and Peyriere [1] that $(\varphi_n)_{n=0}^{\infty}$ is a basis in the Hardy spaces $H^p(\boldsymbol{T})$, $1 < p < \infty$, in the following sense. With the inner product defined for two functions $f,\ g$ on the unit circle $\boldsymbol{T}$ as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta,$$

we have that the MT series converges in $H^p$, i.e.

$$\sum_{n=0}^{N}\langle f,\varphi_n\rangle\varphi_n \xrightarrow{N\to\infty} f. \quad (1)$$

**The non-linear phase unwinding.** There is a natural connection between the MT series and the non-linear phase unwinding decomposition introduced in the dissertation of Nahon [2]. For a function $F$ in $H^p(\boldsymbol{T})$ one can consider the Blaschke factorization

$$F(z) - F(0) = B(z)F_1(z),$$

where $B$ is a Blaschke product and $F_1$ is in $H^p(\boldsymbol{T})$ and does not have zeros in the unit disc. Iterating the procedure, one obtains the formal unwinding series

$$F = F(0)B + F_1(0)BB_1 + \cdots + F_n(0)BB_1 \dots B_n + \cdots.$$

Numerical simulations from [2] suggest that the right-hand side of the above equation converges back to the function and generally this happens at exponential rate. The result of Coifman and Peyri\'ere implies convergence in $H^p(\boldsymbol{T})$. The case $p = 2$ was previously obtained by Qian [3], who had developed a similar theory to phase unwinding independently of Nahon in [4]. Coifman and Steinerberger [5] proved convergence in several different contexts including convergence in fractional Sobolev spaces $H^s$, $s > -\frac{1}{2}$, if the initial function $F$ is in $H^{s+\frac{1}{2}}$.

If at each step of the unwinding decomposition the Blaschke product $B_n$ has finitely many zeros, for example if $F$ is holomorphic in an $\varepsilon$-neighborhood of the unit disk, we can consider the sequence of all zeros o$B, B_1, \dots$. The associated MT series will then reproduce the unwinding decomposition. Intuitively, making the zeros adapted to the function should accelerate the convergence. For this reason, the MT series is also called the Adaptive Fourier Transform.

For an overview of these constructions, we refer to the recent paper [6]. For some further results and intuition on the unwinding decomposition we refer to [7, 8].

**Results.**We are interested in almost everywhere convergence of the MT series (1). By standard techniques, almost everywhere convergence can be deduced from estimates of the maximal partial sum operator. Denote

$$Tf(e^{ix}) := T^{(a_n)}f(e^{ix}) := sup_N|\sum_{n=0}^{N}\langle f,\varphi_n\rangle\varphi_n(e^{ix})|.$$

In fact, we show that up to a Hilbert transform and a maximal function, the operator $T$ is equal to

$$T^{(a_n)}f(e^{ix}) := sup_N|\int_{-\pi}^{\pi} f(e^{iy})B_n(e^{iy})^{-1}\frac{dy}{\sin\frac{x-y}{2}}|. \quad (2)$$

**Question:** Is the maximal operator (8) bounded on $L^p$?

If $a_n \equiv 0$, then the MT series reduces to the classical Fourier series and the operator (2) reduces to the Carleson operator. In this case the positive answer to the above question is given by the Carleson – Hunt theorem [9, 10].

We give two partial answers to this question in this paper. First, if the points are in a compact disc inside the open unit disc, the problem becomes a more benign perturbation of the Carleson – Hunt theorem. In this case, we quantify the $L^p$ norm of the maximal partial sum operator depending on the distance of the compact disc from the unit circle.

**Theorem 1.** *Let* $0 < r < 1$ *and let* $(a_n)_{n=1}^{\infty}$ *be an arbitrary sequence such that* $|a_n| \leq r$ *for all* $n$*. Then, for* $1 < p \leq 2$,

$$\|T^{(a_n)}\|_{L^p(\boldsymbol{T})\to L^p(\boldsymbol{T})} \leq C(p)\log\frac{1}{1-r}. \tag{3}$$

*For* $2 \leq p < \infty$*, we have the better estimate*

$$\|T^{(a_n)}\|_{L^p(\boldsymbol{T})\to L^p(\boldsymbol{T})} \leq C(p)\sqrt{\log\frac{1}{1-r}}. \tag{4}$$

Furthermore, for $1 < p \leq 2$, we have a lower bound in the sense, that for every $0 < r < 1$ there exists a sequence $(a_n)_{n=1}^{\infty}$ with $|a_n| \leq r$ such that

$$\|T^{(a_n)}\|_{L^p(\boldsymbol{T})\to L^p(\boldsymbol{T})} > C(p)\sqrt{\log\frac{1}{1-r}}. \tag{5}$$

In particular, the bounds (4) and (5) are sharp for $p = 2$.

There is a conformally invariant version of Theorem 1 for $p = 2$ and arbitrary compact sets inside the disc. In that case, the quantity $\log\frac{1}{1-r}$ is replaced by the hyperbolic diameter of the compact set. Let us denote by $m_b(z) := \frac{z-b}{1-\bar{b}z}$ the Mobius transform taking $b$ to 0.

**Proposition.** Let $(a_n)_{n=1}^{\infty}$ and $b$ be points in the unit disk, then

$$\|T^{(a_n)}\|_{L^2(\boldsymbol{T})\to L^2(\boldsymbol{T})} = \|T^{(m_{-b}(a_n))}\|_{L^2(\boldsymbol{T})\to L^2(\boldsymbol{T})}.$$

Furthermore, if $1 < q < p < \infty$, then

$$\|T^{(m_{-b}(a_n))}\|_{L^p(\boldsymbol{T})\to L^p(\boldsymbol{T})} \leq \delta(q,p)\|T^{(a_n)}\|_{L^q(\boldsymbol{T})\to L^q(\boldsymbol{T})}, \tag{6}$$

where $\delta(q,p) > 0$ are some constants that blow up as $q$ and $p$ get closer.

For the inequality (6), sparse domination bounds for Carleson-like operators from [11] are used. They allow to pass from the boundedness of $T^{(a_n)}$ for one $p_0$ to the boundedness for all $p \geq p_0$. Thus, (6) implies a symmetric qualitative statement: $T^{(a_n)}$ is bounded on $L^p$ for all $p > r$ if and only if $T^{(m_{-b}(a_n))}$ is bounded on $L^p$ for all $p > r$ . Ideally, one might expect also the symmetric quantitative result

$$\|T^{(m_{-b}(a_n))}\|_{L^p(\boldsymbol{T})\to L^p(\boldsymbol{T})} \sim \|T^{(a_n)}\|_{L^q(\boldsymbol{T})\to L^q(\boldsymbol{T})},$$

however, we do not know how to prove or disprove it.

In Theorem 1, we first obtain the logarithmic dependence (3) for all exponents. We prove a bound independent of $r$ for small scales by the techniques of the polynomial Carleson theorem, discussed in the next subsection, and we apply a triangle inequality for the large scales. Then, for $p = 2$ we are able to improve the estimate for the large scales by using a $TT^*$argument and the analyticity of Blaschke products. The bound (4) for $p > 2$ follows by black-boxing sparse domination results for Carleson-type opeators such as Theorems 9.1 and 9.2 in [11]. Whether (3) and (4) are sharp we do not know.

We turn to the second partial answer, the case when the points are in a non-tangential approach region to the boundary.

**Theorem 2.** *Let $a_n$ be inside the triangle with vertices $(1,0), (\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$ for all $n$, then*

$$\|T \quad\|_{L^2(\boldsymbol{T}) \to L^2(\boldsymbol{T})} \leq C.$$

Theorem 2 is true for similar non-tangential approach regions to other points on the circle. Furthermore, if one takes the union of $k$ approach regions for $k$ distinct points at once, then it is possible to prove along the lines of Theorem 2 that the $L^2(\boldsymbol{T})$ norm of the operator (2) is bounded by $k$. If the boundary points for the approach regions are chosen to be equidistant, then the construction giving (5) also provides the lower bound $\sqrt{\log k}$. The sharp bound for this configuration is again unknown to us. It could also be interesting to consider approach regions to countably many points for various configurations.

**Connection to polynomial Carleson theorem.** Theorem 1 and Theorem 2 turn out to be closely related to the polynomial Carleson theorem. Let us recall a special case of the polynomial Carleson operator [12,13,14]

$$C_d f(x) := sup_Q sup_{0<\varepsilon<N} \left| \int_{\varepsilon<|x-y|<N} f(y) e^{iQ(y)} \frac{dy}{x-y} \right|,$$

where the first supremum is taken over polynomials of degree at most $d$. The case $d = 1$ is the classical Carleson operator. Its weak $L^2$ bounds were implicit in Carleson's paper [8] on almost everywhere convergence of the Fourier series, Hunt improved this to $L^p$ bounds, $1 < p < \infty$. Alternative approaches appeared in Fefferman [15], Lacey and Thiele [16]. On the other hand, Stein and Wainger [17] proved the case $d \geq 2$ but restricted to polynomials without the linear term. Lie combined the two techniques in [12,13] to prove the general $L^p$ bounds for $C_d$. Finally, Zorin – Kranich generalized the argument to higher dimensions and non-convolution Calderon – Zygmund kernels in [14]. For a gentle introduction to Carleson's theorem and a discussion of the different approaches we refer to Demeter's paper [18].

We reformulate the polynomial Carleson theorem as Theorem 3 describing the general necessary conditions on the set of the phase functions. Then, Theorem 1 and Theorem 2 are proved by showing that the Blaschke phases satisfy these conditions and applying Theorem 3.

Let $K$ be a translation-invariant Calderon–Zygmund kernel on $\boldsymbol{R}$ with the associated operator bounded on $L^2(\boldsymbol{R})$. We assume that there exists an $\varepsilon > 0$ such that $supp\, K$ lies in $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$. Let $\mathbf{Q}$ be a countable subset of $C^2(\boldsymbol{R})$. For each interval $I$ and $P, Q \in \mathbf{Q}$ we define

$$d_I(P, Q) := sup_{x,y \in I} |(P - Q)(x) - (P - Q)(y)|,$$

and assume that $d_I$ is a metric on $\mathbf{Q}$.

We impose the following conditions on $\mathbf{Q}$. Assume that there exists a constant $C > 0$ such that

**A.** (the analog of Lemma 2.6 in [14]) for any intervals $J, I$ with $J$ lying in $I$ and $|I| \leq \varepsilon$, and for $P, Q \in \mathbf{Q}$ we have

$$d_I(P,Q) \le C\frac{|I|}{|J|}d_J(P,Q), \qquad d_J(P,Q) \le C\frac{|J|}{|I|}d_I(P,Q);$$

**B.** (John-Ellipsoid Property) for any $\gamma > 1$ and interval $I$ with $|I| \le \varepsilon$, any $(\gamma, d_I)$-ball can be covered by $C\gamma$ number of $(1, d_I)$--balls;

**C.** (the analog of Lemma A.1 in [14], oscillatory integral estimate) for any measurable function $g: \boldsymbol{R} \to \boldsymbol{C}$, interval $J$ with $supp\ (g)$ lying in $J$ and $|J| \le \varepsilon$, and any $P, Q\ \in \mathbf{Q}$ we have

$$|\int_J e^{i(P-Q)(x)} g(x)dx|$$

$$\le Csup_{|y|<(1+d_J(P,Q))^{-1}|J|}\int_{\boldsymbol{R}} |g(x)-g(x-y)|dx.$$

**Theorem 3.** *Assume conditions A,B and C hold for the set* $\mathbf{Q}$. *We define the operator* $T: L^1_{loc}(\boldsymbol{R}) \to L^0(\boldsymbol{R})$ *as*

$$Tf(x) := sup_{Q\in\mathbf{Q}} sup_{0<\varepsilon<N}\left|\int_{\varepsilon<|x-y|<N} f(y)e^{iQ(y)}K(x-y)dy\right|.$$

*Further, let* $0 < \alpha < \frac{1}{2}$ *and* $0 < \mu < \rho < \infty$. *Let* $F, G$ *be measurable subsets of* $\boldsymbol{R}$ *and* $\tilde{F} := \{M\mathbf{1}_F > \mu\}$, $\tilde{G} := \{M\mathbf{1}_G > \rho\}$. *Then the following inequalities hold(with the implicit constants depend on* $C$ *and* $\alpha$, *but are independent of* $\varepsilon$*)*

$$\|T\|_{L^2(\boldsymbol{T})\to L^2(\boldsymbol{T})} \le 1,$$
$$\left\|\mathbf{1}_G T \mathbf{1}_{\boldsymbol{R}\setminus\tilde{G}}\right\|_{L^2(\boldsymbol{T})\to L^2(\boldsymbol{T})} \le \mu^\alpha,$$
$$\left\|\mathbf{1}_{\boldsymbol{R}\setminus\tilde{F}} T \mathbf{1}_{\boldsymbol{F}}\right\|_{L^2(\boldsymbol{T})\to L^2(\boldsymbol{T})} \le \rho^\alpha.$$

Theorem 3 is a version on Zorin – Kranich's Theorem 1.5 in [14]. Lemma B.1 in the latter establishes $L^p$ boundedness of $T$, for $1 < p < \infty$, from the last two inequalities.

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## On Almost-Everywhere Convergence of Malmquist–Takenaka Series

We prove $L^p$ bounds for the maximal partial sum operator of the Malmquist-Takenaka series under additional assumptions on the zeros of the Mobius transforms. We locate the problem in the time-frequency setting and, in particular, we connect it to the polynomial Carleson theorem.

**Գ. Տ. Մնացականյան**

**Մալմքուիստ – Տակենակայի շարքի համարյա ամենուրեք զուգամիտության մասին**

Մալմքուիստ – Տակենակայի համակարգի մասնակի գումարների մաքսիմալ օպերատորի համար ապացուցված են $L^p$ գնահատականներ Մոբյուսի ձևափոխության զրոների վրա լրացուցիչ պայմանների դեպքում: Խնդիրը տեղակալված է տարածա-հաճախային վերլուծման համատեքստում և մասնավորապես կապվում է Կարլեսոնի բազմանդամային թեորեմի հետ:

**Г. Т. Мнацаканян**

**О почти везде сходимости ряда Малмквиста – Такенаки**

Доказаны $L^p$ оценки для максимального оператора частичных сумм ряда Малмквиста – Такенаки при дополнительных предположениях о нулях преобразований Мёбиуса. Проблема рассмотрена в частотно-временной постановке и, в частности, связана с полиномиальной теоремой Карлесона.

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