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A Hybrid Control Problem for a Linear System with Constant Coefficients

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Problem Description. Assume we have a state space model which have the following dynamics

$$\dot{x}_i = a_{i1}x_1 + \ldots + a_{in}x_n + p_{i1}y_1 + \ldots + p_{ik}y_k + b_{i1}u_1 + \ldots + b_{ir}u_r , \qquad (1)$$

$$\dot{y}_j = c_{j1}y_1 + \ldots + c_{jk}y_k + d_{j1}x_1 + \ldots + d_{jm}x_m \qquad (2)$$

where the coefficients $a_{il}, b_{is}, c_{jq}, d_{jf}, p_{iq}$ are real constants and , $i = 1,\ldots,n$ $j = 1,\ldots,k$, $r \le n-m,$, $m \le k \le n$ $l = 1,\ldots,n$, $s = 1,\ldots,r$, $q = 1,\ldots,k$, . $f = 1,\ldots,m$ Also $x_1,\ldots,x_n, y_1\ldots,y_k$ are the states of the system and $u_1,\ldots,u_r$ is the control actions applied to the system.

Let us introduce the below notations.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad P = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \vdots & \vdots \\ p_{n1} & \cdots & p_{nk} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nr} \end{pmatrix},$$

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \vdots & \vdots \\ c_{k1} & \cdots & c_{kk} \end{pmatrix} \quad D = \begin{pmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \vdots & \vdots \\ d_{k1} & \cdots & d_{km} \end{pmatrix}.$$

Thus, we can rewrite the system (1)-(2) as a system of matrix equations as shown below.

$$\dot{x} = Ax + Py + Bu \,, \tag{3}$$

$$\dot{y} = Cy + D\overline{x} \,. \tag{4}$$

Here, $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$ is an n dimensional column vector, $y = \begin{pmatrix} y_1 & \cdots & y_k \end{pmatrix}^T$ is a k dimensional column vector, $\overline{x} = \begin{pmatrix} \overline{x}_1 & \cdots & \overline{x}_m \end{pmatrix}^T$ is an m dimensional column vector that contains some m states of x and $u = \begin{pmatrix} u_1 & \cdots & u_r \end{pmatrix}^T$ is an r dimensional column vector.

Let us now define the following problem.

Problem Definition 1. We are given the system (1)-(2) (or (3)-(4)), the time period $[t_0, t_1]$, the initial position of some of the states (maximum number of the states of (3) can be $n/2$ and for (4) the number can be k of the system $\left(x(t_0);\ y(t_0)\right) = \left(x_0;\ y_0\right)$ and the desired final position of some of the states (maximum number of the states of (3) can be $n/2$ and for (4) the number can be k of the system $x(t_1) = x_1$. It is required to find the control inputs $u(t), (t_0 \le t \le t_1)$ such that it drives the system from its given initial position to its desired final position.

Assume, the matrices A, B, C, D are such that

$$rank\, K_1 = \left\{D, CD, \ldots, C^{k-1}D\right\} = k \tag{5}$$

and

$$rank\, K_2 = \left\{B_1, A_1B_1, \ldots, A_1^{n+k-1}B_1\right\} = n + k \,. \tag{6}$$

Where A_1 is the following $(n+k)\times(n+k)$ matrix

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{1m} & a_{1m+1} & \cdots & a_{1n} & p_{11} & \cdots & p_{1k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & a_{nm+1} & \cdots & a_{nn} & p_{n1} & \cdots & p_{nk} \\ d_{11} & \cdots & d_{1m} & 0 & \cdots & 0 & c_{11} & \cdots & c_{1k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{k1} & \cdots & d_{km} & 0 & \cdots & 0 & c_{k1} & \cdots & c_{kk} \end{pmatrix}$$

and B_1 is an $(n+k)\times r$ matrix as shown below.

$$B_1 = \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nr} \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Suppose, also, that there is an additional condition for the system (2) (or (4)) which assumes that the states $y_1, \ldots, y_k$ remain close to the point $O(0 \ \cdots \ 0)$, $y(t_1) = y_1$ is infinitely close to zero and there is a constraint given on the system (2) (or (4)). Now suppose that the constraint is given as

$$J[\bullet] = \int_{t_0}^{\infty} \left(\sum_{i,j=1}^{k} \alpha_{ij} y_i y_j + \sum_{i,j=1}^{m} \beta_{ij} x_i x_j \right) dt . \tag{7}$$

Thus, we can choose $x_1, \ldots, x_m$ to be control actions for the system (2) (or (4)) and hence, we can define to following problem.

Problem Definition 2. Assume we are given the dynamics of the state space model (2) (or (4)) and the constraint (7). We need to find the control actions $\overline{x}_1^0[t], \ldots, \overline{x}_m^0[t]$ such that the system (2) (or (4)) becomes asympto-tically stable and the constraint (7) reaches its minimal value.

Now, because of the assumption (5) the system (2) (or (4) becomes fully controllable [1], hence, for any reasonable initial position $y(t_0) = y_0$ there exists unique $\left(x_1^0 \ \cdots \ x_m^0\right)^T$ column vector of control actions which solve the problem 2 [2]. This means that also the states $y_1^0(t), \ldots, y_k^0(t)$ will be calculated uniquely, moreover

$$\lim_{t \to \infty} \overline{x}_i^0[t] = 0, \ (i = 1, \ldots, m) \tag{8}$$

and

$$\lim_{t \to \infty} y_i^0(t) = 0, \ (i = 1, \ldots, k) . \tag{9}$$

Now that we solved the second problem, we will discuss the problem 1. So, by substituting the functions $\overline{x}_1^0[t], \ldots, \overline{x}_m^0[t]$ and $y_1^0(t), \ldots, y_k^0(t)$, which we gained by solving the problem 2, into the system (1) (or (3)), we can rewrite the system as

$$\dot{x}_i = a_{i\,m+1} x_{m+1} + \ldots + a_{i\,n} x_n + b_{i1} u_1 + \ldots + b_{i\,n} u_n + f_i(t) \tag{10}$$

where $i = m+1, \ldots, n$, and

$$f_i(t) = a_{i1} \overline{x}_1^0[t] + \ldots + a_{i\,m} \overline{x}_m^0[t] + p_{i1} y_1^0(t) + \ldots + p_{i\,k} y_k^0(t) \tag{11}$$

where $i = 1, \ldots, n$.

It is obvious that first m equations from the system (10) will become algebraic equations because the functions $x_1^0[t],\ldots x_m^0[t]$ will be already known.

According to (6) the system (1)(2) is filly controllable, hence, it remains to calculate the control actions $u=\left(u_1(t) \quad \cdots \quad u_r(t)\right)^T$ which solve the first problem, and that can be done by choosing some known algorithm. Thus, the problem is solved.

Numerical Example. Now let as demonstrate how the above theory works by bringing an example of a system.

Suppose we have the controllable system (12) and (13) [3]:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{m}{m+m_p} g x_8 - \frac{m_p}{m+m_p}\frac{g}{l_p} y_1, \qquad \dot{x}_3 = x_4,$$
$$\dot{x}_4 = -\frac{m}{m+m_p} g x_7 - \frac{m_p}{m+m_p}\frac{g}{l_p} y_3, \qquad \dot{x}_5 = x_6, \tag{12}$$
$$\dot{x}_7 = x_{10}, \qquad \dot{x}_8 = x_{11}, \qquad \dot{x}_9 = x_{12},$$

$$\dot{x}_6 = u_1, \quad \dot{x}_{10} = u_2 - \frac{g}{I_{xx}} y_3, \quad \dot{x}_{11} = u_3 - \frac{g}{I_{yy}} y_1, \quad \dot{x}_{12} = u_4, \tag{13}$$

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = -g x_8 - \frac{g}{l_p} y_1, \qquad \dot{y}_3 = y_4, \qquad \dot{y}_4 = g x_7 - \frac{g}{l_p} y_3 \tag{14}$$

For the systems (12), (13) and (14) we can check the conditions (5) and (6). We will have:

$$rank\ K_1 = 4 \text{ and } rank\ K_2 = 16.$$

In this example the vector $\overline{x}$ will be $\overline{x} = \left(\overline{x}_1[t] \ \ \overline{x}_2[t]\right)^T = \left(x_7 \ x_8\right)^T$. So, we will need to solve the second problem for the system (14) and find $\overline{x}$. The constraint (7) in this example will take the following form:

$$J[\bullet] = \int_0^\infty \left(x_{14}^2 + x_{16}^2 + u_5^2 + u_6^2\right) d\tau. \tag{15}$$

This system describes the motion of an Unmanned Aerial vehicle with pendulum hanging from it.

So, for this system we have

$$A=\begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\frac{m}{m+m_p}g&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-\frac{m}{m+m_p}g&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \quad P=\begin{pmatrix} 0&0&0&0\\ -\frac{g}{l_p}&0&0&0\\ 0&0&0&0\\ 0&0&-\frac{g}{l_p}&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&-\frac{g}{I_{xx}}&0\\ -\frac{g}{I_{yy}}&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$B=\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \quad C=\begin{pmatrix} 0&1&0&0\\ -\frac{g}{l_p}&0&0&0\\ 0&0&0&1\\ 0&0&-\frac{g}{l_p}&0 \end{pmatrix} \quad D=\begin{pmatrix} 0&0\\ 0&-g\\ 0&0\\ g&0 \end{pmatrix}$$

To solve the above-mentioned problem, we choose to use Bellman-Lyapunov method. Thus, we will have

$$\begin{aligned} \bar{x}_1[t]=x_7(t)&=-\frac{0.5\sqrt{g}}{\sqrt{l_p}\sqrt{-4+gl_p}}\cdot\left[\exp(\lambda_- t)-\exp(\lambda_+ t)\right],\\ \bar{x}_2[t]=x_8(t)&=\frac{0.5\sqrt{g}}{\sqrt{l_p}\sqrt{-4+gl_p}}\cdot\left[\exp(\lambda_- t)-\exp(\lambda_+ t)\right]. \end{aligned} \quad (16)$$

Were the exponential coefficients are

$$\lambda_-=\left(-gl_p-\sqrt{g}\sqrt{l_p}\sqrt{-4+gl_p}\right)\Big/2l_p\ ,$$

$$\lambda_{+} = \left(-gl_p + \sqrt{g}\sqrt{l_p}\sqrt{-4+gl_p}\right)/2l_p ,$$

One can simply check that (8) and (9) are correct for this example. Now, what we need to do, is to substitute (16) into (14) and integrate (14) for some initial conditions. For this example we have chosen the initial conditions as:

$$y_1(0) = 0.5,\ y_2(0) = 0,\ y_3(0) = 0.5,\ y_4(0) = 0 .$$

After integrating the system (14) we can substitute $\overline{x}_1, \overline{x}_2, y_1, y_2, y_3, y_4$ back into (12), (13) and integrate the system. This way we will have all the state trajectories and control actions except for x_5, x_6, x_9, x_{12}. The reason is that those states can be decoupled from (12), (13) and, also, the corresponding equations are not dependent of (14) at all. So, to get those states we can proceed in a few different ways. As for example, one can formulate an optimal control problem, or just assign some realistic path for those states to follow. Hence, the two problems defined in the theoretical part of this paper will be completely solved.

The resulting control actions for (12), (13) are the following:

$$u_2(t) = \frac{0.5\sqrt{g}\left(\lambda_{-}^2 \cdot \exp\left(e^{\lambda_{+}t}\right) - 0.25\lambda_{+}^2 \cdot \exp\left(e^{\lambda_{-}t}\right)\right)}{\sqrt{l_p}\sqrt{-4+gl_p}} + \\ + \frac{0.25g\left(2l_p\lambda_{+} \cdot \exp(\lambda_{+}t) + 2l_p\lambda_{-} \cdot \exp(\lambda_{-}t)\right)}{I_{xx}\sqrt{-4+gl_p}}$$

Conclusion. A hybrid control problem of a system of linear differential equations with constant coefficients is discussed in this paper. It was assumed that some of the states of the system have to satisfy some additional conditions. To ensure those conditions are satisfied, some of the states of one subsystem were chosen to be additional control actions in second subsystem. Then, an optimal stabilization problem was defined and solved for the second subsystem using Lyapunov-Bellman method. The special states which were chosen to be control actions and the corresponding optimal trajectories were acquired for the second subsystem. Afterwards, those solutions are substituted in the first subsystem and the main control problem was solved. An example of a hybrid control problem for a system was presented where the system represents the dynamics of a UAV carrying a pendulum. The specially chosen states are indicated and were calculated as optimally stabilizing control actions, after which the main control actions of the system we gained and shown by the end of the example.

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A Hybrid Control Problem for a Linear System with Constant Coefficients

The problem of hybrid control of a linear system movement is discussed. It is assumed that the coefficients of the considered system are constants. Also, it is assumed that a subsystem of our main plant satisfies some conditions. Then, some of the states in that subsystem are considered as control inputs and thus the subsystem is optimally stabilized using LQR stabilizer. The optimal solutions of the subsystem are used as control actions on the whole system. A numerical example system is presented for which the hybrid of optimal stabilizing control actions of a subsystem and control actions for the main system is introduce.

ՀՀ ԳԱԱ թղթակից անդամ Արա Ս. Ավետիսյան, Ա. Ս. Շահինյան

Հաստատուն գործակիցներով գծային համակարգի հիբրիդային ղեկավարման խնդիրը

Քննարկվում է գծային համակարգի շարժման հիբրիդային կառավարման խնդիրը: Ենթադրվում է, որ դիտարկվող համակարգի գործակիցները հաստատուններ են: և մեր հիմնական կայանի ենթահամակարգը բավարարում է որոշ պայմանների. այդ ենթահամակարգի որոշ դիրքեր համարվում են որպես հսկիչ մուտքեր, և ենթահամակարգը օպտիմալորեն կայունացվում է՝ օգտագործելով LQR կայունացուցիչ: Ենթահամակարգի օպտիմալ լուծումներն օգտագործվում են որպես ղեկավարման ազդակներ ամբողջ համակարգի վրա: Ներկայացված է թվային համակարգի ղեկավարման օրինակ, որի դեպքում ներդրվում է ենթահամակարգի օպտիմալ կայունացնող ազդակների և հիմնական համակարգի ղեկավարման ազդակների հիբրիդը:

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Задача гибридного управления линейной системой с постоянными коэффициентами

Обсуждается проблема гибридного управления движением линейной системы. Предполагается, что коэффициенты рассматриваемой системы постоянны и подсистема нашего основного агрегата удовлетворяет некоторым условиям: некоторые состояния в этой подсистеме рассматриваются как управляющие входы, и, таким образом, подсистема оптимально стабилизируется с помощью стабилизатора LQR. Оптимальные решения подсистемы используются как управляющие воздействия на всю систему. Представлен численный пример системы, для

которой вводится гибрид управляющих воздействий, исходящих из оптимальной стабилизации подсистемы, и управляющих воздействий на основную систему.

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