ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱ
НАЦИОНАЛЬНАЯ АКАДЕМИЯ НАУК АРМЕНИИ
NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ДОКЛАДЫ ԶԵԿՈՒՅՑՆԵՐ REPORTS

Հատոր Том Volume 121 2021 № 1

MATHEMATICS

УДК 510.64

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On Some Properties of Several Proof Systems for 2-valued Propositional Logic

(Submitted by academician Yu. H. Shoukourian 9/II 2021)

Keywords: *minimal tautology; Frege systems; substitution Frege systems; Sequent proof systems; proof complexity measures; monotonous systems.*

1. Introduction. The minimal tautologies, i.e. tautologies, which are not a substitution of a shorter tautology, play a main role in the proof complexity area. Really all hard propositional formulas, proof complexities of which are investigated in many well-known papers, are minimal tautologies. There is a traditional assumption that minimal tautology must be no harder than any substitution in it. This idea was revised at first by Anikeev in [1]. He had introduced the notion of monotonous proof system and had given two examples of no complete propositional proof systems: monotonous system, in which the proof lines of all minimal tautologies are no more, than the proof lines for results of a substitutions in them, and no monotonous system, the proof lines of substituted formulas in which can be less than the proof lines of corresponding minimal tautologies. We introduce for the propositional proof systems the notions of monotonous by lines and monotonous by sizes of proofs. In [2] it is proved that Frege systems F no monotonous neither by lines nor by size. In this paper we prove that substitution Frege systems SF, the well-known propositional sequent systems PK, PK^- and corresponding systems with substitution rule SPK**,** $SPK^-$are no monotonous neither by lines nor by size.

This work consists of 4 main sections. After Introduction we give the main notion and notations as well as some auxiliary statements in Preliminaries. The main results are given in the last two sections.

2. **Preliminaries.** We will use the current concepts of a propositional formula, a classical tautology, sequent, Frege proof systems, sequent systems for classical propositional logic and proof complexity [3]. Let us recall some of them.

2.1. The considered systems of 2-valued propositional logic. Following [3] we give the definition of main systems, which are considered in this point.

2.1.1. A Frege system F uses a denumerable set of 2-valued variables, a finite, complete set of propositional connectives; F has a finite set of inference rules defined by a *figure* of the form $\frac{A_1 A_2 \ldots A_m}{B}$(the rules of inference with zero hypotheses are the axioms schemes); F must be sound and complete, i.e. for each rule of inference $\frac{A_1 A_2 \ldots A_m}{B}$every truth-value assignment, satisfying $A_1 A_2 \ldots A_m$, also satisfies B, and F must prove every tautology.

The particular choice of a language for presented propositional formulas is immaterial in this consideration. However, because of some technical reasons we assume that the language contains the propositional variables p, q and p_i ,q_i $(i \geq 1)$, logical connectives $\neg, \supset, \vee, \wedge$ and parentheses (,). Note that some parentheses can be omitted in generally accepted cases. We assume also that Fhas well known inference rule *modus ponens.*

2.1.2 A substitution Frege system SF consists of a Frege system F augmented with the substitution rule with inferences of the form $\frac{A}{A\sigma}$for any substitution $\sigma = \left(\varphi_{i_1}\ \varphi_{i_2} \ldots \varphi_{i_s}\ p_{i_1}\ p_{i_2} \ldots p_{i_s}\right)$, s $\geq$ 1, consisting of a mapping from propositional variables to propositional formulas, and $A\sigma$ denotes the result of applying the substitution to formula A, which replaces each variable in A with its image under σ. This definition of substitution rule allows the simultaneous substitution of multiple formulas for multiple variables of A without any restrictions.

2.1.3. PK^{-}system uses the denotation of sequent$\Gamma \to \Delta$ where Γ is antecedent and Δ is succedent.

The axioms of PK^{-} system are

1) $p \to p$, 2) $\to T$,

where p is propositional variable and Tdenotes «truth».

For every formulas A ,B and for any sequence of formulas Γ, Δ the logic rules are.

$$\supset\to \frac{\Gamma\to\Delta,\ A \quad B,\ \Gamma\to\Delta}{A\supset B,\ \Gamma\to\Delta} \qquad \to\supset \frac{A,\ \Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\supset B}$$

$$\vee\to \frac{A,\ \Gamma\to\Delta \quad B,\ \Gamma\to\Delta}{A\vee B,\ \Gamma\to\Delta} \qquad \to\vee \frac{\Gamma\to\Delta,\ A}{\Gamma\to\Delta,\ A\vee B} \text{ and } \to\vee \frac{\Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\vee B}$$

$$\wedge\to \frac{A,\ \Gamma\to\Delta}{A\wedge B,\ \Gamma\to\Delta} \text{ and } \wedge\to \frac{B,\ \Gamma\to\Delta}{A\wedge B,\ \Gamma\to\Delta} \qquad \to\wedge \frac{\Gamma\to\Delta,\ A \quad \Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\wedge B}$$

$$\neg\to \frac{\Gamma\to\Delta,\ A}{\neg A,\ \Gamma\to\Delta} \qquad \to\neg \frac{A,\ \Gamma\to\Delta}{\Gamma\to\Delta,\ \neg A}$$

The only structured inference rule is

$\frac{\Gamma\to\Delta}{\Gamma'\to\Delta'}$ Str.r., where $\Gamma'(\Delta')$ contains $\Gamma(\Delta)$ as a set.

2.1.4. The system PK is PK^{-} augmented with cut-rule

$$\frac{\Gamma_1\to\Delta_1,\ \ A \qquad A,\ \Gamma_2\to\Delta_2}{\Gamma_1,\Gamma_2\to\Delta_1,\Delta_2}$$

2.1.5. Substitution sequent calculus $SPK(SPK^{-})$ defined by adding to PK **(**PK^{-}**)** the following rule of substitution

$$\frac{\Gamma\to\Delta,\ \ A\ (p)}{\Gamma\to\Delta,\ \ A\ (B)},$$

where simultaneous substitution of the formula $B$is allowed for the variable p, and where p does not appear in Γ, Δ.

Note (1). Let $\Gamma\to\Delta$ be some sequent, where Γ is a sequence of formulas $A_1, A_2, \dots, A_l$ $(l \geq 0)$ and $\Delta$is a sequence of formulas $B_1, B_2, \dots, B_m$ $(m \geq 0)$. The formula form of sequent $\Gamma\to\Delta$ is the formula $\varphi_{\Gamma\to\Delta}$, which is defined usually as follows:

1) $A_1 \wedge A_2 \wedge \dots \wedge A_l \supset B_1 \vee B_2 \vee \dots \vee B_m \quad l, m \geq 1$
2) $A_1 \wedge A_2 \wedge \dots \wedge A_l \ \supset\perp$ for $l \geq 1$, $m = 0$ $\perp$is false
3) $B_1 \vee B_2 \vee \dots \vee B_m$ for $l = 0$ and $m \geq 1$.

2.2. Proof complexity measures. By $|\varphi|$ we denote the size of a formula φ, defined as the number of all logical signs in it. It is obvious that the full size of a formula, which is understood to be the number of all symbols is bounded by some linear function in $|\varphi|$.

In the theory of proof complexity two main characteristics of the proof are: *t*- complexity (length), defined as the number of proof steps, l-complexity (size), defined as sum of sizes for all formulas in proof (size) [3].

Let ϕ be a proof system and φ be a tautology. We denote by $t_{\varphi}^{\phi}\left(l_{\varphi}^{\phi}\right)$the minimal possible value of *t*-complexity (l-complexity) for all ϕ-proofs of tautology φ.

Definition 2.2.1. A tautology is called ***minimal*** if it is not a substitution of a shorter tautology. We denote by S(φ) the set of all formulas, every of which is result of some substitution in a minimal tautology φ.

Definition 2.2.2. The proof system ϕ is called ***t-monotonous*** **(**l-***monotonous*****)** if for every minimal tautology φ and for all formulas $\psi \in S(\varphi)$ $t_{\varphi}^{\phi} \leq t_{\psi}^{\phi}$ $(l_{\varphi}^{\phi} \leq l_{\psi}^{\phi})$.

Definition 2.2.3. Sequent $\Gamma\to\Delta$ is called **minimal valid** if its formula form $\varphi_{\Gamma\to\Delta}$ is minimal tautology.

Definition 2.2.4. Sequent proof system Φ is called *t-monotonous* (l-*monotonous*) if for every valid sequent $\Gamma\to\Delta$ and for every sequent $\Gamma_1\to\Delta_1$ such that $\varphi_{\Gamma_1\to\Delta_1}\in$ S($\varphi_{\Gamma\to\Delta}$) $t_{\Gamma\to\Delta}^{\phi} \leq t_{\Gamma_1\to\Delta_1}^{\phi}$ ($l_{\Gamma\to\Delta}^{\phi} \leq l_{\Gamma_1\to\Delta_1}^{\phi}$).

2.3. Essential subformulas of tautologies

For proving the main results we use the notion of *essential subformulas*, introduced in [4].

Let F be some formula and $Sf(F)$be the set of all non-elementary subformulas of formula F.

For every formula F, for every $\varphi \in Sf(F)$ and for every variable p by F_φ^p is denoted the result of the replacement of the subformulas φ everywhere in F by the variable p. If $\varphi \notin Sf(F)$, then F_φ^p is F.

We denote by $Var(F)$ the set of all variables in F.

Definition 2.3.1. Let p be some variable that $p \notin Var(F)$ and $\varphi \in Sf(F)$ for some tautology F. We say that φ is an ***essential subformula*** in F iff F_φ^p is non-tautology.

The set of essential subformulas in tautology F we denote by *Essf(F)*, the number of essential subformulas by *Nessf(F)* and the sum of sizes of all essential subformulas by *Sessf(F).*

If F is minimal tautology, then $Essf(F) = Sf(F)$

Definition 2.3.2. The subformula φ is essential for valid sequent $\Gamma \rightarrow \Delta$ if it is essential for its formula form.

In [4] the following statement is proved.

Proposition 1. Let F be a tautology and $\varphi \in Essf(F)$, then

a) in every F-proof of F subformula φ must be essential either at least in some axiom, used in proof or in formula $A_1 \supset (A_2 \supset (\ldots \supset A_m)\ldots) \supset B$ for some used in proof inference rule $\frac{A_1 A_2 \ldots A_m}{B}$,

b) in every SF-proof of F, where $\frac{A}{A_{\sigma_1}}$, $\frac{A}{A_{\sigma_2}}$, … ,$\frac{A}{A_{\sigma_k}}$ (k≥1) are used substitution rules, subformula φ must be essential either at least in some axiom, used in proof, or in formula $A_1 \supset (A_2 \supset (\ldots \supset A_m)\ldots) \supset B$ for some used in proof inference rule $\frac{A_1 A_2 \ldots A_m}{B}$, and or must be result of successive substitutions σ_{i_1}, σ_{i_2}, …, σ_{i_r} for $1 \leq i_1, i_2, \ldots, i_r \leq k$ in them.

Note (2) that for every Frege system the number of essential subformulas both in every axiom and in formula $A_1 \supset (A_2 \supset (\ldots \supset A_m)\ldots) \supset B$ for every inference rule $\frac{A_1 A_2 \ldots A_m}{B}$ is bounded with some constant.

Note (3). It is not difficult to prove that all above statements are true for every formula form of valid sequent and axioms and rules of above mentioned sequent systems.

2.4. Main Formulas. It is known, that in the alphabet, having 3 letters, for every $n > 0$ a word with size n can be constructed such, that neither of its subwords repeats in it twice one after the other [5].

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be some of such words in the alphabet $\{a, b, c\}$. In [4] the formulas ψ_n are obtained as follow:

For $n > 0$ let $\psi_{n+1,n} = (p_0 \supset \neg\neg p_0)$. Let the formula $\psi_{i+1,n}$ for the subwords

$\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n (1 \leq i \leq n)$ be constructed then:

1) if $\alpha_i = a$ then $\psi_{i,n} = (p_i \supset p_i) \wedge \psi_{i+1,n}$;

2) if $\alpha_i = b$ then $\psi_{i,n} = (\neg p_i \vee p_i) \supset \psi_{i+1,n}$;

3) if $\alpha_i = c$ then $\psi_{i,n} = (\neg p_i \wedge p_i) \vee \psi_{i+1,n}$;

As formula $\psi_n(p_0, p_1, p_2, \dots, p_n)$we take the formula $\psi_{1,n}(p_0, p_1, p_2, \dots, p_n)$, for which we have $|\psi_n| = \Theta(n)$ and all subformulas $\psi_{i,n} (1 \leq i \leq n+1)$are essential for ψ_n, therefore *Nessf*(ψ_n) $= \Omega(n)$ and *Sessf*(ψ_n) $= \Omega(n^2)$.

Note (4). As neither of these essential subformulas of formulas $\psi_n$cannot be obtained from other by substitution rule, it is proved in [4] that formulas $\psi_n$require more or equal than n steps and n^2 size both in the Frege systems and substitution Frege systems. It is obvious, that this statement is valid for sequent $\rightarrow \psi_n$ for the all mentioned sequent systems also.

3. Main results. Here we give the main theorem, but at first we must give the following easy proved auxiliary statements.

Lemma 1. *a) The minimal **t-complexity** and l-**complexity** of Frege proofs and thus of substitution Frege proofs for formula $p \supset p$ are bounded by constant.*

*b) For each formulae A and B the minimal **t-complexity** of Frege proofs and thus of substitution Frege proofs for formula $A \supset (\neg A \supset B)$ is bounded by constant, just as its minimal l-complexity is bounded by c·max($|A|$, $|B|$) for some constant c.*

Lemma 2. *a) If F is minimal tautology and p is some variable that $p \notin Var(F)$, then the formula $p \supset F$ is also minimal tautology, and all essential subformulas of formula F are essential for formula $p \supset F$.*

b) If $\rightarrow F$ is minimal valid sequent and p is some variable that $p \notin Var(F)$ then sequent $\rightarrow p \supset F$ is also minimal valid sequent, and all essential subformulas of sequent $\rightarrow F$ are essential for sequent $\rightarrow p \supset F$.

Theorem. *Every Frege system F, substitution Frege system SF, sequent systemsPK, PK^-, SPK, $SPK^-$are neither t-monotonous nor l-**monotonous.***

Proof. Let us consider the tautologies

$\varphi_n = p \supset \psi_n$, $\alpha_n = \neg(p \supset p) \supset \psi_n$ and $\beta_n = (p \supset p) \supset \alpha_n$,

where $\psi_n$are the formulas from previous section and variable p is not belong to *Var*(ψ_n). Note that **for every *n* formula α_n (sequent $\rightarrow \alpha_n$) belongs to S(φ_n) (S($\rightarrow \varphi_n$)).** According to the statement of Lemma 2. every formula φ_n is minimal tautology and the sequent$\rightarrow \varphi_n$ is minimal valid sequent.

For every *n* the formula α_n can be deduced in every Frege system F as follows: at first we deduce formula $p \supset p$, then $\beta_n = (p \supset p) \supset \alpha_n$and lastly $\alpha_n$by modus ponens. From statement of Lemma 1 we obtain for lengths and sizes of proofs in every Frege system for the set of formulas α_n the bounds $O(1)$ and $O(n)$ accordingly. It is obvious that for more "stronger" substitution systems Frege the bounds are no more.

By statements of Lemma 2 and Note (4) we obtain for lengths and sizes of proofs in every Frege system and substitution systems Frege for the set of formulas $\varphi_n$the bounds $\Omega(n)$ and $\Omega(n^2)$ accordingly.

Now we give deduction of the sequent $\rightarrow\neg\ (p\supset p)\ \supset\psi_n$in the system PK^{-}as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{p\rightarrow p}{\rightarrow(p\supset p)}}{\neg(p\supset p)\rightarrow}}{\neg(p\supset p)\rightarrow\psi_n}}{\rightarrow\neg\ (p\supset p)\ \supset\psi_n}.$$

It is obvious, that the bounds for lengths and sizes of proofs in system $PK^{-}$for the set of sequents $\rightarrow\alpha_n$ are $O(1)$ and $O(n)$ accordingly, therefore for more "stronger" systems PK, SPK,SPK^{-} the bounds are no more.

By statements of Lemma 2 and Note (4) we obtain for lengths and sizes of proofs in the systems PK, PK^{-},SPK,SPK^{-} for the set of sequents $\rightarrow\varphi_n$the bounds $\Omega(n)$ and $\Omega(n^2)$ accordingly.

Remark. Above results only for Frege systems are parts of publications [6, 7].

I am grateful to my supervisor, professor of YSU AnahitChubaryan for her encouragement and fruitful discussions, and also for helpful suggestions, which improved and expanded the results.

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On Some Properties of Several Proof Systems for 2-valued Propositional Logic

In this work we investigate the relations between the proofs complexities of minimal tautologies and of results of substitutions in them in some systems of 2-valued classical propositional logic. We show that the result of substitution can be proved easier, than corresponding minimal tautology, therefore the systems, which are considered in this paper, are no monotonous neither by lines nor by size.

Գ. Վ. Պետրոսյան

Երկարժեք ասույթային տրամաբանության որոշակի արտածման համակարգերի հատկությունները

Հետազոտված է մինիմալ նույնաբանությունների և նրանցում տեղադրություն-ների արտածման բարդությունների միջև հարաբերությունը որոշակի երկարժեք ասույթային դասական տրամաբանության համակարգերում: Ցույց է տրվել, որ տեղա-դրման արդյունք հանդիսացող բանաձևերը կարող են արտածվել ավելի հեշտ, այդ իսկ պատճառով դիտարկված համակարգերը մոնոտոն չեն:

Г. В. Петросян

О некоторых свойствах нескольких систем доказательства 2-значной логики высказываний

Исследована связь между сложностями доказательств минимальных тавтологий и результатами подстановок в них в некоторых системах 2-значной классической логики высказываний. Показано, что результат подстановки доказывается проще, чем соответствующая минимальная тавтология, поэтому рассматриваемые в статье системы не являются монотонными ни по линиям, ни по размеру.

References

1. *Cook S. A., Reckhow A. R.* – Journal of Symbolic logic. 1979. V. 44. P. 36-50.
2. *Аникеев А. С.* – Мат. заметки. 1972. Т. 11. Вып. 2. С. 165-174.
3. *Чубарян Ан.* – В сб.: Мат. вопр. кибернетики. Вып. 14. М. Физматлит. 2005. С. 49-56.
4. *Адян С. И.* Проблемы Бернсайда и тождества в группах. М. Наука. 1975. 335 с.
5. *Chubaryan An., Petrosyan G.* – Evolutio. Естественные науки. 2016. Вып. 3. С. 12-14.