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On a Class of Integro-Difference Equations

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1. Statement of the problem. Let μ_k, m_k $(k = 1, \dots, N)$ be positive numbers such that $\mu_k \neq \mu_j$ for $k \neq j$. The system of linear equations

$$\varphi_j(x) + \sum_{i=1}^{N} \frac{m_i m_j e^{-(\mu_i+\mu_j)x}}{\mu_i + \mu_j} \varphi_i(x) = m_j e^{-\mu_j x}, \ \ j = 1, \dots, N$$

uniquely determines infinitely differentiable functions $\varphi_1, \dots, \varphi_N$ satisfying the conditions $e^{\mu_k|x|}\varphi_k \in L_\infty(\mathbb{R})$, $k = 1, \dots, N$ (see [1-3]). Note that the numbers $-\mu_k^2$ and the functions φ_k $(k = 1, \dots, N)$ form complete systems of eigenvalues and corresponding eigenfunctions of a certain Sturm-Liouville operator with a reflectionless potential. Reflectionless potentials are connected with a family of explicit solutions of the Korteweg–de Vries equation, the so-called $\mathcal{N}$-soliton solutions (see [2]).

The set of all almost periodic functions of the form

$$b(x) = \sum_{j=-\infty}^{\infty} \beta_j e^{i\nu_j x} \ \ (x \in \mathbb{R}) \tag{1.1}$$

where $\nu_j \in \mathbb{R}$, $\beta_j \in \mathbb{C}$ $(j \in \mathbb{Z})$ and $\beta_i \neq \beta_j$ for $i \neq j$, taken with the norm

$$\|b\|_{\mathrm{APW}} := \sum_{j=-\infty}^{\infty} |\beta_j|,$$

is a Banach algebra which will be denoted by APW (see [4]).

Let

$$\hat{k}(\lambda) = (Fk)(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} k(t) dt$$

be the Fourier transform of a function $k \in L_1(\mathbb{R})$. $W_0(\mathbb{R})$ will denote the Banach algebra $\{Fk: k \in L_1(\mathbb{R})\}$ with the norm $\|Fk\|_{W_0(\mathbb{R})} := \|k\|_{L_1(\mathbb{R})}$. The set of functions $A := \{a = b + \hat{k} : b \in \mathrm{APW}, \mathrm{k} \in \mathrm{L}_1(\mathbb{R})\}$, taken with the norm $\|a\|_A := \|b\|_{\mathrm{APW}} + \|\hat{k}\|_{W_0(\mathbb{R})}$, is a Banach algebra and coincides with the direct sum of the algebras APW and $W_0(\mathbb{R})$.

Let $a = b + \hat{k} \in A$, $k \in L_1(\mathbb{R})$, and let $b \in \mathrm{APW}$ be given by (1.1). We define the operators $T_0(a), T_1(a), T(a): L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+)$ ($\mathbb{R}_+ = (0, \infty), 1 \leq p \leq \infty$) by the formulas

$$(T_0(a)y)(x) := \sum_{k=-\infty}^{\infty} \beta_k y(x - \nu_k) + \int_0^{\infty} k(x-t) y(t) dt\,,$$

$$(T_1(a)y)(x) := \sum_{j=1}^{N} \varphi_j(x) \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} k(\tau) e^{\mu_j \tau \operatorname{sgn}(x-\tau-t)} d\tau \right\} \varphi_j(t) y(t) dt +$$

$$+ \sum_{k=-\infty}^{\infty} \beta_k \sum_{j=1}^{N} e^{\mu_j \nu_k} \varphi_j(x) \int_{x-\nu_k}^{\infty} \varphi_j(t) y(t)\, dt +$$

$$+ \sum_{k=-\infty}^{\infty} \beta_k \sum_{j=1}^{N} e^{-\mu_j \nu_k} \varphi_j(x) \int_0^{x-\nu_k} \varphi_j(t) y(t)\, dt,$$

$$T(a) := T_0(a) - T_1(a),$$

where we assume that $y(x) = 0$ for $x < 0$.

$T_0(a)$ is a Wiener-Hopf operator with a symbol a. This fact makes it possible to find criteria for invertibility and one-sided invertibility of the operator $T_0(a)$ and to describe its kernel and cokernel. In this work we will present analogous results for the operator $T(a)$ which is not a Wiener-Hopf operator, but has properties close to those of $T_0(a)$. The function a will be also called the symbol of the operator $T(a)$.

2. Factorization of the symbol. The mean value

$$M(e^{-\lambda x} b) := \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} e^{-\lambda x} b(x) dx$$

of the function $e^{-\lambda x} b$, where $b \in \mathrm{APW}$ is given by (1.1), equals β_j if $\lambda = \nu_j$ and vanishes if $\lambda \neq \{\nu_j : j \in \mathbb{Z}\}$. Therefore the Bohr-Fourier spectrum $\Omega(b) := \{\lambda \in \mathbb{R} : M(e^{-\lambda x} b) \neq 0\}$ of the function b coincides with the set $\{\nu_j : j \in \mathbb{Z}\}$. Let APW^+ (APW^-) denote the subalgebra of all functions $b \in \mathrm{APW}$ satisfying the inclusion $\Omega(b) \subset [0, \infty)$ ($\Omega(b) \subset (-\infty, 0]$).

Every function $b \in \mathrm{APW}$ satisfying the condition

$$\inf_{\lambda \in \mathbb{R}} |b(\lambda)| > 0 \tag{2.1}$$

has a right APW factorization

$$b(\lambda) = b_-(\lambda) e^{i\lambda \varkappa_b} b_+(\lambda) \tag{2.2}$$

where $\varkappa_b \in \mathbb{R}$, $b_-^{\pm 1} \in \mathrm{APW}_-$, $b_+^{\pm 1} \in \mathrm{APW}_+$ (see [4], [7]). The number $\varkappa_b$ is called the *mean motion* or the *almost periodic index* of the function b and can be computed by the formula

$$\varkappa_b = \lim_{\ell \to \infty} \frac{1}{2\ell} \left[(\arg b)(\ell) - (\arg b)(-\ell) \right], \tag{2.3}$$

where $\arg b$ is to be understood as an arbitrary continuous function on $\mathbb{R}$, satisfying the equality $b = |b| \exp(i \arg b)$.

The function $e^{-i\lambda\varkappa_b} b(\lambda)$ has a representation of the form

$$e^{-i\lambda\varkappa_b} b(\lambda) = e^{\psi(\lambda)} \quad (\lambda \in \mathbb{R})$$

with $\psi \in \mathrm{APW}$, i.e., the logarithm $\psi(\lambda) = \log(e^{-i\varkappa_b \lambda} b(\lambda))$ exists and can be written as

$$\psi(x) = \sum_{k=-\infty}^{\infty} \psi_k e^{i\lambda_k x} \quad (x \in \mathbb{R})$$

where λ_k $(k \in \mathbb{Z})$ are distinct real numbers and ψ_k $(k \in \mathbb{Z})$ are nonzero complex numbers satisfying the condition

$$\sum_{k=-\infty}^{\infty} |\psi_k| < \infty .$$

The functions $b_\pm$ in (2.2) can be chosen in the following way:

$$b_-(x) = \exp\left(\sum_{\lambda_k < 0} \psi_k e^{i\lambda_k x} \right), \qquad b_+(x) = \exp\left(\sum_{\lambda_k \geq 0} \psi_k e^{i\lambda_k x} \right).$$

Let $S: L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be the singular integral operator defined by the formula

$$(Sy)(t) \coloneqq \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{y(s)}{s - t}\, ds$$

where the integral is to be understood in the Cauchy principal value sense, and let $P_\pm = \frac{1}{2}(I \pm S)$. Then

$$(x + i) P_+ \left(\frac{1}{x + i}\, \psi \right) = \sum_{\lambda_k \geq 0} \psi_k e^{i\lambda_k x} + \sum_{\lambda_k < 0} \psi_k e^{\lambda_k} ,$$

$$(x + i) P_- \left(\frac{1}{x + i}\, \psi \right) = \sum_{\lambda_k < 0} \psi_k e^{i\lambda_k x} - \sum_{\lambda_k < 0} \psi_k e^{\lambda_k}$$

(see [4]). Since the functions $b_\pm$ are determined up to a constant multiple, we may choose

$$b_\pm(x) = \exp\left((x + i) P_\pm \left(\frac{1}{x + i}\, \psi \right) \right).$$

Let

$$W(\mathbb{R}) \coloneqq \mathbb{C} + W_0(\mathbb{R}) = \{ c + Fk : c \in \mathbb{C}, k \in L_1(\mathbb{R}) \}$$

be the Wiener algebra on $\mathbb{R}$. $W(\mathbb{R})$ is a Banach algebra with the norm $\|c+Fk\| := |c| + \|k\|_{L_1(\mathbb{R})}$ (see [4]).

Consider also the algebras

$$W^{\pm}(\mathbb{R}) = \{c + Fk : c \in \mathbb{C}, k \in L_1(\mathbb{R}), k(x) = 0 \text{ for } \pm x < 0\}.$$

Every function $d \in W(\mathbb{R})$ satisfying the condition

$$\inf_{\lambda \in \mathbb{R}} |d(t)| > 0 \tag{2.4}$$

has a Wiener-Hopf factorization in the algebra $W(\mathbb{R})$, i.e., it has a representation of the form

$$d(x) = d_-(x)(r(x))^{\varkappa_d} d_+(x), \tag{2.5}$$

where $d_-^{\pm 1} \in W^-(\mathbb{R})$, $d_+^{\pm 1} \in W^+(\mathbb{R})$, $\varkappa_d \in \mathbb{Z}$ and $r(x) := (x-i)/(x+i)$. The integer $\varkappa_d$ is unique and can be computed by the formula

$$\varkappa_d = \frac{1}{2\pi}\big(\arg d(+\infty) - \arg d(-\infty)\big). \tag{2.6}$$

The function $r^{-\varkappa_d} d$ has a logarithm in $W(\mathbb{R})$, i.e., there exist $c_0 \in \mathbb{R}$ and $g \in L_1(\mathbb{R})$ such that $r^{-\varkappa_d} d = \exp(c_0 + \hat{g})$.

The functions d_- and d_+ in (2.5) can be determined by the formulas

$$d_{\pm} = \exp\left((x+i)P_{\pm}\left(\frac{1}{x+i}(c_0 + \hat{g}(x))\right)\right).$$

Moreover, the equalities

$$(x+i)P_+\left(\frac{1}{x+i}\,(c_0 + \hat{g}(x))\right) = c_0 + \int_{-\infty}^{0} e^s k(s)ds + F(\chi_+ g)(x),$$

$$(x+i)P_-\left(\frac{1}{x-i}\,(c_0 + \hat{g}(x))\right) = -\int_{-\infty}^{0} e^s k(s)ds + F(\chi_- g)(x)$$

hold, where χ_+ (χ_-) is the characteristic function of the set $\mathbb{R}_+$ ($\mathbb{R}_- := (-\infty, 0)$) (see [4]). The last two formulas, together with the fact, that $d_{\pm}$ are determined up to a constant multiple, show that the functions $d_{\pm}$ can also be determined by the equalities

$$d_+ = \exp[c + F(\chi_+ q)], \qquad d_- = \exp[F(\chi_- g)].$$

Consider the subalgebras $A_{\pm} := \mathrm{APW}^{\pm} + W^{\pm}(\mathbb{R})$ of the algebra A. It is known that every function $a \in A$ satisfying the condition

$$\inf_{x \in \mathbb{R}} |a(\lambda)| > 0 \tag{2.8}$$

has a factorization of the form

$$a(x) = a_-(x) e^{i\varkappa_b x} \big(r(x)\big)^{\varkappa_d} a_+(x) \tag{2.9}$$

with $\varkappa_b \in \mathbb{R}$, $\varkappa_d \in \mathbb{Z}$, $a_+^{\pm 1} \in A_+$ and $a_-^{\pm 1} \in A_-$ (see [7]).

Assume that the condition (2.8) is satisfied for the function $a = b + \hat{k}$ where $b \in \mathrm{APW}$ and $k \in L_1(\mathbb{R})$. Since (2.8) implies (2.1) (see [7]), hence the function b is invertible in APW. Decompose a into the product $a = bd$ where $d = 1 + b^{-1}\hat{k}$. Since $W_0(\mathbb{R})$ is an ideal of the algebra A, hence $d \in W(\mathbb{R})$.

(2.8) implies the condition (2.4), too. It follows that the numbers $\varkappa_b$ and $\varkappa_d$ in (2.9) are uniquely determined by the formulas (2.3) and (2.6); the functions $a_\pm$ are uniquely determined by the formulas $a_\pm = b_\pm d_\pm$, (2.2) and (2.5).

The next theorem reveals the fundamental importance of the condition (2.8) in the behavior of the operator $T(a)$.

Theorem 2.1. *Let $a \in A$. The operator $T(a)$ is normally solvable if and only if the condition (2.8) is satisfied.*

3. Main results. Define the operators $\mathcal{K}_1, \mathcal{K}_2 : L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+)$, $1 \le p < \infty$ by the formulas

$$(\mathcal{K}_1 y)(x) = y(x) + \sum_{k=1}^{N} m_k e^{-\mu_k x} \int_x^\infty \varphi_k(\tau) y(\tau) d\tau ,$$

$$(\mathcal{K}_2 y)(x) = y(x) + \sum_{k=1}^{N} m_k \varphi_k(x) \int_x^\infty e^{-\mu_k \tau} y(\tau) d\tau .$$

From now on, the condition (2.8) is assumed to be satisfied; the numbers $\varkappa_\mathrm{b}$, $\varkappa_\mathrm{d}$ and the functions $\mathrm{a}_\pm$ are assumed to be determined by (2.9). Note that $\mathrm{r} \in \mathrm{W}(\mathbb{R}) \subset \mathrm{A}$ and the operator $\mathrm{T}_0(\mathrm{r}^\mathrm{k})$ $(\mathrm{k} \in \mathbb{Z})$ coincides with the Wiener-Hopf operator with a symbol r^k (see [7]). Furthermore it is assumed that the operator $\mathrm{T(a)}$ acts in the space $\mathrm{L_p}(\mathbb{R}_+)$, $1 \le \mathrm{p} < \infty$ and the equation

$$T(a)y = f \qquad (3.1)$$

is considered in the same space.

Theorem 3.1. *If $\varkappa_b > 0$, then the operator $T(a)$ is left invertible. In order that the equation (3.1) be solvable, it is necessary and sufficient that the following conditions be satisfied:*

a) The function $T_0(a_-^{-1})\mathcal{K}_1 f$ vanishes on the interval $[0, \varkappa_b]$ for $\varkappa_d \ge 0$. Moreover, if $\varkappa_d > 0$, then

$$\int_0^\infty t^k e^{-t} (T(a_-^{-1})\mathcal{K}_1 f)(t) dt = 0 , \qquad k = 0, \dots, \varkappa_d - 1 . \qquad (3.2)$$

b) For $\varkappa_d < 0$, the restriction of the function $e^t (T_0(r^{-\varkappa_d}) T_0(a_-^{-1})\mathcal{K}_1 f)(t)$ to $[0, \varkappa_d]$ is a polynomial of degree $-\varkappa_d - 1$.

Theorem 3.2. *If $\varkappa_b < 0$, then the operator $T(a)$ is left invertible. For $\varkappa_d \ge 0$, the kernel of $T(a)$ consists of all functions of the form*

$$\mathcal{K}_2 T_0(a_+^{-1}) T_0(r^{-\varkappa_d}) g ,$$

where g is an arbitrary function in $L_p(\mathbb{R}_+)$, vanishing on the interval $(-\varkappa_b, \infty)$ and satisfying the additional conditions

$$\int_0^\infty g(t)\, t^j e^{-t}\, dt = 0, \qquad j = 0, \dots, \varkappa_d - 1$$

for $\varkappa_d > 0$.

For $\varkappa_d < 0$*, the kernel of* $T(a)$ *consists of all functions of the form*

$$\mathcal{K}_2 T_0(a_+^{-1})(g+q),$$

where g *is an arbitrary function in* $L_p(\mathbb{R}_+)$*, vanishing on the interval* $(-\varkappa_b, \infty)$*, and* q *is a polynomial of degree at most* $-\varkappa_d - 1$.

Theorem 3.3. *Let* $\varkappa_b = 0$.

a) The operator T(a) is invertible for $\varkappa_d = 0$ *and*

$$\left(T(a)\right)^{-1} = \mathcal{K}_2 T_0(a_+^{-1}) T_0(a_-^{-1}) \mathcal{K}_1.$$

b) For $\varkappa_d > 0$*, the operator* $\mathcal{K}_2 T_0(a_+^{-1}) T_0(r^{-\varkappa_d}) T_0(a_-^{-1}) \mathcal{K}_1$ *is a left inverse of* $T(a)$*, and equation (3.1) is solvable if and only if conditions (3.2) are satisfied.*

c) For $\varkappa_d < 0$*, the operator* $\mathcal{K}_2 T_0(a_+^{-1}) T_0(r^{-\varkappa_d}) T_0(a_-^{-1}) \mathcal{K}_1$ *is a right inverse of* $T(a)$*, and the kernel of* $T(a)$ *consists of all functions of the form* $\mathcal{K}_2 T_0(a_+^{-1}) q$*, where* q *is a polynomial of degree at most* $-\varkappa_d - 1$.

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On a Class of Integro-Difference Equations

We consider a class of integro-difference equations which, by their solvability properties, are close to the Wiener-Hopf equation with the symbol given as the sum of an almost periodic function expanding in an absolutely convergent Fourier series and a Fourier transform of the function summable on the whole axis.

Հ. Ա. Ասատրյան, Ա. Հ. Քամալյան, Մ. Ի. Կարախանյան

Ինտեգրալատարբերականային հավասարումների մի դասի մասին

Դիտարկվում է ինտեգրալատարբերակային հավասարումների դաս, որոնք լուծելիության հատկություններով մոտ են Վիներ-Հոպֆի հավասարմանը, որի սիմվոլը ներկայացվում է Ֆուրիեի բացարձակ զուգամետ շարքով, համարյա պարբերական ֆունկցիայի և առանցքի վրա հանրագումարելի ֆունկցիայի Ֆուրիեի ձևափոխության գումարի տեսքով:

А. А. Асатрян, А. Г. Камалян, М. И. Караханян

Об одном классе интегрально-разностных уравнений

Рассматривается класс интегрально-разностных уравнений, близких по свойствам разрешимости к уравнению Винера – Хопфа, символ которого представляется в виде суммы почти периодической функции, разлагающейся в абсолютно сходящийся ряд Фурье и преобразования Фурье суммируемой на оси функции.

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