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On Two Extremal Problems

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Let X be a normed space, $\{f_k\}_1^n$ $(n=\dim X\ge 2)$ be a set of linear independent elements from X. Denote by $\Pi_n = \left\{z : z\in\mathbb{C}^n : \max\limits_{1\le k\le n}|z_k|\le 1\right\}$ the unit ball (cube) in $\mathbb{C}^n$ and by $\Sigma_n = \left\{z : z\in\mathbb{C}^n : \max\limits_{1\le k\le n}|z_k| = 1\right\}$ – the unit sphere.

Problem 1. We seek

$$\sup_{\{\alpha_k\}\in\Pi_n}\|\alpha_1 f_1+\alpha_2 f_2+\cdots+\alpha_n f_n\| \tag{1}$$

and the element from X maximizing norm in (1).

A similar problem is considered in [1], where the exact convergence rate of an iterative method of solution of a Hilbert space operator equation is calculated.

As the mapping $p(\{\alpha_k\}) = \|\alpha_1 f_1+\alpha_2 f_2+\cdots\alpha_n f_n\|$ is continuous and Π_n is compact, the element M realizing the maximum in (1) exists. Easy to see that p is positive homogeneous and subadditive, i.e. $p(c\alpha)=|c|\,p(\alpha)$, $p(\alpha+\beta)\le p(\alpha)+p(\beta)$.

To describe the element M we first prove some auxiliary propositions. Let H_n be the Hemming cube $[0;1]^n$.

Proposition 1. The set of extreme points of H_n consists of the vertices of H_n.

Proof. Let $x\in H_n$ and for some i the strict inequality $0<x_i<1$ be satisfied. Then there exists a positive number ε such that $\varepsilon \le x_i \le 1-\varepsilon < 1$. Denote by $x_{\pm}$ the element obtained from x changing x_i by $x_i \pm \varepsilon$. Evidently $x = 0.5\left(x_+ + x_-\right)$, meaning that x cannot be an extreme point. If y is a vertex of H_n, i.e. consists of 0 and, 1 and $y = ta + (1-t)b,\ t \in [0;1]$, then $a = b = y$.

According to the Caratheodory theorem ([2], Ch.1, §1, P.1.6) any element of a convex compact subset C of $\mathbb{R}^n$ may be represented as a convex combination of $n+1$ extreme points of C.

Proposition 2. Any element of Π_n is a convex combination of at most $2n+2$ extreme points of Π_n.

Proof. Let $z = \left\{r_1 e^{i\alpha_1}, r_2 e^{i\alpha_2}, \cdots, r_n e^{i\alpha_n}\right\} \in \Pi_n$. As $r = \left(r_1, r_2, \cdots, r_n\right) \in H_n$ by Caratheodory's theorem $r = t_1 e^1 + t_2 e^2 + \cdots + t_{n+1} e^{n+1}$, where t_kr $0, \sum_{k=1}^{n+1} t_k = 1$ and $\left\{e^k\right\}_1^{n+1}$ are the extreme points of H_n. Then

$$r_1 = t_1 e_1^1 + t_2 e_1^2 + \cdots + t_{n+1} e_1^{n+1}, r_2 = t_1 e_2^1 + t_2 e_2^2 + \cdots + t_{n+1} e_2^{n+1},$$
$$\cdots, r_n = t_1 e_n^1 + t_2 e_n^2 + \cdots + t_{n+1} e_n^{n+1} \qquad (2)$$

Multiplying the k-th equality of (2) by $exp\left(i\alpha_k\right)$, we get $z = t_1 g^1 + t_2 g^2 + \cdots + t_{n+1} g^{n+1}$, where $g_k^j = e_k^j exp\left(i\alpha_k\right)$. Replacing (if the necessity arises) the extreme points of H_n by a pair of the extreme points of Π_n, we complete the proof.

Example 1. The extreme points of H_2 are $e^1(0;0), e^2(1;0), e^3(0;1)$ and $e^4(1;1)$.

For a point $P(a;b) \in H_2$ we have the representation $t_1 = 1 - \max\{a,b\}$, $t_2 = a - \min\{a,b\}$, $t_3 = b - \min\{a,b\}$, $t_4 = \min\{a,b\}$.

Let $z_1 = r_1 e^{i\alpha_1}, z_2 = r_2 e^{i\alpha_2}$. Then

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left(1-r_2\right)\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \left(r_2 - r_1\right)\begin{pmatrix} 0 \\ e^{i\alpha_2} \end{pmatrix} + r_1 \begin{pmatrix} e^{i\alpha_1} \\ e^{i\alpha_2} \end{pmatrix},\ if\ r_2 > r_1$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left(1-r_1\right)\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \left(r_1 - r_2\right)\begin{pmatrix} e^{i\alpha_1} \\ 0 \end{pmatrix} + r_2 \begin{pmatrix} e^{i\alpha_1} \\ e^{i\alpha_2} \end{pmatrix},\ if\ r_1 > r_2.$$

Finally we get (if $r_2 > r_1$)

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1-r_2}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1-r_2}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{r_2-r_1}{2}\begin{pmatrix} 1 \\ e^{i\alpha_2} \end{pmatrix} + \frac{r_2-r_1}{2}\begin{pmatrix} -1 \\ e^{i\alpha_2} \end{pmatrix} + r_1\begin{pmatrix} e^{i\alpha_1} \\ e^{i\alpha_2} \end{pmatrix}.$$

Theorem 1. Let $M = \sum_{k=1}^{n} z_k f_k$ *be the solution of the Problem 1. Then* $|z_k| = 1$ *for any* $k, k = 1, 2, \cdots, n.$

Proof. By (1) $\|M\| = p(z)\mathrm{b}\sum_k t_k \|g^k\| \le \|M\|$. As the inequality in fact is equality, condition $\|g^k\| < 1$ implies t_k=0 so there exists at least one index j such that $\|g^j\| = \|M\|$.

Remark 1. The condition in this theorem is not sufficient. In $L^2(0;1)$ we have $\|1-x\| = 1/\sqrt{3}$ and $\|1+x\| = \sqrt{7/3}$.

The solution, in general, is not unique. For an orthonormal set $\{f_k\}_1^n$ and any set of real numbers $\{\varphi_k\}_1^n$ the equality $\left\|\sum_{k=1}^{n} e^{i\varphi_k} f_k\right\|^2 = n$ is satisfied.

Consider now the case, where the norm in X is generated by an inner product. In this case we denote $X = H$. We have

$$\|M\|^2 = \sum_{k,m=1}^{n} \langle f_k, f_m \rangle \alpha_k \overline{\alpha}_m.$$

If condition of Theorem 1 is satisfied, then

$$\|M\|^2 = \sup_{|\xi_1|=|\xi_2|=\cdots|\xi_n|=1} \sum_{k,m=1}^{n} \langle f_k, f_m \rangle \xi_k \overline{\xi}_m.$$

Example 2. Find the solution of Problem 1 in the space $L^2(0;1)$ taking $\{f_k\} = \{1, x, \cdots, x^{n-1}\}$.

We have $\langle f_k, f_m \rangle = \dfrac{1}{k+m-1}$, arriving at the Hilbert matrix

$$D_n = \begin{pmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ . & . & . & . \\ 1/n & 1/(n+1) & \cdots & 1/(2n-1) \end{pmatrix}.$$

Denoting by S_n the sum of all elements of D_n, we have the recurrences

$$S_{n+1} = S_n + 2\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \frac{1}{2n+1}.$$

Below are first eight values of
S_n $-1, 7/3, 37/10, 533/105, 1627/252, 67/353, 2951/320, 2853/269.$
Finally $\|M\|^2 = S_n$ and $M(x) = 1 + x + \cdots + x^{n-1}$.

Now we pass to the second problem and seek

$$\inf_{\{\alpha_k\}\in\Sigma_n} \|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\| \quad (3)$$

Let $a, b \in X$ and the function $g : \mathbb{C} \to \mathbb{R}^+$ be defined by the formula $g(\lambda) = \|a - \lambda b\|$.

Recall that b is said to be orthogonal to a if $\|a - \lambda b\| \, \mathrm{r} \, \|a\|$ for any $\lambda \in \mathbb{C}$. In a unitary space (where the norm is defined by an inner product) this orthogonality coincides with the orthogonality with respect to the inner product.

Theorem 2. *Let* $m = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$ *be the solution of (3) and* $|\alpha_k| < 1$. *Then* f_k *is orthogonal to* m.

Proof. For any $\lambda \in \mathbb{C}$ and $t \in [0;1]$ we have

$$g((1-t)\lambda + t\mu) = \|a - ((1-t)\lambda + t\mu)b\| =$$
$$\|(1-t)a - (1-t)\lambda b + ta - t\mu b\| \le (1-t)\|a - \lambda b\| +$$
$$+ t\|a - \mu b\| = (1-t)g(\lambda) + tg(\mu),$$

implying that the function g is convex.

Let $0 < |\lambda| < 1 - |\alpha_k|$, then $|\alpha_k - \lambda| < |\lambda| + |\alpha_k| < 1$. The minimality of $\|m\|$ implies $\|m\| \le \|m - \lambda f_k\|$, meaning that 0 is the local minimum point for the function $\|m - \lambda f_k\|$. As it is well known ([3], Ch. 1, 3.3, Theorem 5) for any convex function defined on a convex set any local minimum is the global minimum, so $\|m\| \le \|m - \lambda f_k\|$ for any $\lambda \in \mathbb{C}$, completing the proof.

The estimate of the norm of the linear combination of basis elements in a Hilbert space from below is proposed in [4]. According to Theorem 1 of [4] for any set $\{\alpha_k\} \subset \mathbb{C}$ the following inequality holds

$$\left\|\sum_{k=1}^{n} \alpha_k f_k\right\| \, \mathrm{r} \, d_{n-1}(E_n, H) \cdot \max_{1\le k\le n} |\alpha_k|,$$

where $d_{n-1}(E_n, H)$ is the $(n-1)$- th Kolmogorov width of the linear span in

H of $\{f_k\}_1^n$ with coefficients belonging to Π_n.

By Lemma 1 of [5]

$$d_{n-1}(E_n, H) = \frac{1}{\max\limits_{1\le k\le n}\|f_k^*\|}$$

where $\{f_k^*\}_1^n$ is the biorthogonal with $\{f_k\}$ set, i.e. $\langle f_k, f_j^*\rangle = \delta_{jk}$ and $\{f_k^*\}_1^n$ lie in the linear subspace generated by $\{f_k\}_1^n$.

Denote by G the Gram matrix

$$G = \begin{pmatrix} \langle f_1, f_1\rangle & \langle f_1, f_2\rangle & \cdots & \langle f_1, f_n\rangle \\ \langle f_2, f_1\rangle & \langle f_2, f_2\rangle & \cdots & \langle f_2, f_n\rangle \\ . & . & . & . \\ \langle f_n, f_1\rangle & \langle f_n, f_2\rangle & \cdots & \langle f_n, f_n\rangle \end{pmatrix}.$$

As it is well known the inverse matrix G^{-1} is generated by the biorthogonal set $\{f_k^*\}$. Let $\|f_k^*\|^2 = \max(diag(G^{-1}))$. By the Schwarz inequality $|\langle f_j^*, f_k^*\rangle| < \|f_k^*\|^2, j \ne k$. The coefficients of element m having the minimal norm will be the entries of the $k-$th row of G^{-1}, divided by $\|f_k^*\|^2$ and $\|m\| = \frac{1}{\|f_k^*\|}$.

Example 3. The inverse of D_5 is the matrix

$$\begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix},$$

$m = -1/128 + 3/20x - 21/32x^2 + x^3 - 63/128x^4$ and $\|m\|^2 = 1/179200.$

Remark 2. The solution of problem (3) is not unique. For an orthonormal set $\{f_k\}_1^n$ the minimal norm is equal to 1 and is attained on each element f_k.

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On Two Extremal Problems

Two extremal problems for the norm of linear combinations of basis elements are considered. First we describe the linear combination having the greatest possible norm. Next the linear combination having the least norm is characterized.

Լ. Զ. Գևորգյան

Երկու էքստրեմալ խնդիրների մասին

Քննարկվում են երկու էքստրեմալ խնդիրներ՝ կապված բազիսային տարրերի գծային թաղանթին պատկանող տարրի նորմի հետ: Սկզբում նկարագրվում է առավելագույն նորմ ունեցող տարրը: Այնուհետև բնութագրվում է նվազագույն նորմով տարրը:

Л. З. Геворгян

О двух экстремальных задачах

Рассматриваются две экстремальные задачи о нахождении нормы линейной комбинации базисных элементов. Сначала описывается элемент с максимальной нормой. Затем характеризуется элемент с наименьшей нормой.

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