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Orthogonal Decomposition in Omega-Weighted Classes of Functions Subharmonic in the Half-Plane

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1. Introduction. The present paper gives a harmonic, ω-weighted, half-plane analog of Wirtinger's projection theorem [1] (see also [2], p. 150) and its $(1-r)^{\alpha}$-weighted extension by M. M. Djrbashian (see Theorem VII in [3]), which are for holomorphic in $|z|<1$ functions with square integrable moduls. These results are a continuation of the results of [5] in the half-plane. Then, an orthogonal decomposition is found for some classes of functions subharmonic in the upper half-plane, which is similar to the result of [4] in the unit disc.

After a useful remark, we shall introduce the spaces of functions which are to be considered.

Remark 1.1. It is well-known see, eg. [6], Ch. VI) that the Hardy space h^p $(1\le p<+\infty)$ of real, harmonic in the upper half-plane $G^+:=\{z:\operatorname{Im} z>0\}$ functions, which is defined by the condition

$$\|u\|_{h^p}:=\sup_{y>0}\left\{\int_{-\infty}^{+\infty}\left|u\left(x+iy\right)\right|dx\right\}^{1/p}<+\infty$$

is a Banach space becoming a Hilbert space for $p=2$. Since $|u|^p$ is subharmonic in G^+, the results of Ch. 7 in [7] on the equivalent definition of the holomorphic hardy spaces H^p in G^+ have their obvious analogs for h^p. In particular, the space $h^p(1\le p<+\infty)$ coincodes with the set of all functions harmonic in G^+ and such that

$$\|u\|_{h^p}^{P}=\lim_{R\to+\infty}\lim_{y\to+\infty 0}\inf\int_{-R}^{R}\left|u\left(x+iy\right)\right|^p dx<+\infty$$

and for sufficiently small values of $\rho>0$

$$\lim_{R\to+\infty}\inf\frac{1}{R}\int_{\beta}^{\pi-\beta}\left|u\left(\mathrm{Re}^{i\theta}\right)\right|^{p}\left(\sin\frac{\pi(\vartheta-\beta)}{\pi-2\beta}\right)^{1-\pi/\kappa}d\vartheta=0, \tag{1.1}$$

where $\beta=\arcsin\frac{\rho}{R}=\frac{\pi}{2}-\kappa$. Note that due to Holder's inequality, if (1.1) is true for a $p>1$, then it is true also for $p=1$.

Definition 1.1. $\tilde{\Omega}_{\alpha}(-1<\alpha<+\infty)$ is the set of continuous, strictly increasing in $[0,+\infty)$, continuously differentiable in $(0,+\infty)$ functions ω such that $\omega(0)=0$ and $\omega'(x)\succ\prec x^{\alpha}$, $\Delta<x<+\infty$, for some $\Delta>0$.

Definition 1.2. For any $\omega\in\tilde{\Omega}_{\alpha}(-1<\alpha<+\infty)$, $h_{\omega}^{p}(0<p<+\infty)$ is the set of the real, harmonic in the upper half-plane G^{+} functions for which (1.1) is true along with

$$\|u\|_{p,\omega}:=\left\{\int\int_{G^{+}}\left|u(z)^{p}\,d\mu_{\omega}(z)\right|\right\}^{1/p}<+\infty, \tag{1.2}$$

where $d\mu_{\omega}(x+iy)=dx\,d\omega(2y)$.

2. Some properties of the spaces h_{ω}^{p}. First, we prove that the above introduced classes h_{ω}^{p} are Banach spaces.

Proposition 2.1. $h_{\omega}^{p}\left(1\le p<+\infty,\omega\in\tilde{\Omega}_{\alpha},\ \alpha>-1\right)$ is a Banach spaces with the norm (1.2), which forr $p=2$ becomes a Hilbert space with the inner product

$$(u,v)_{\omega}:=\frac{1}{2\pi}\int\int_{G^{+}}u(z)v(z)\,d\mu_{\omega}(z),\qquad u,v\in h_{\omega}^{2}.$$

Proof. Let $L_{\omega}^{p}(1\le p<+\infty)$ be the Banach space of real functions, which is defined solely by (1.2). Then, it suffices to prove that h_{ω}^{p} is a closed subspace of L_{ω}^{p} for any $1\le p<+\infty$, i.e if a sequence $\{u_n\}_1^{\infty}\subset h_{\omega}^{p}$ converges to some $u\in L_{\omega}^{p}$ in the norm of L_{ω}^{p}, then $u\in L_{\omega}^{p}$. To this end, observe that

$$\int_0^{1/2}d\omega(2y)\int_{-\infty}^{+\infty}\left|u_n(x+iy)-u(x+iy)\right|^{p}dx\to 0\ \text{ as }\ n\to\infty.$$

Hence, by Fatou's lemma $\int_0^1 g(t)\,d\omega(t)=0$ for

$$g(2y):=\liminf_{n\to\infty}\int_{-\infty}^{+\infty}\left|u_n(x+iy)-u(x+iy)\right|^{p}dx.$$

As $\omega\in\tilde{\Omega}_{\alpha}$, there exists a sequence $\eta_k\downarrow 0$ such that $\omega(\eta_{k+1})<\omega(\eta_k)$. Introducing the measure $v(E)=V_E\,\omega$, we conclude that $v([\eta_{k+1},\eta_k])>0$ for any $k\ge1$ and obviously $g(t)=0$ in $[\eta_{k+1},\eta_k]$ almost everywhere with respect to the measure v. On the other hand, $u(x+it)\in L^{p}(-\infty,+\infty)$ for almost every $t>0$ with respect to the measure v. Thus, there sequence $y_k\downarrow 0$ such that simultaneously $g(2y_k)=0$ and $u(x+iy_k)\in L^{p}(-\infty,+\infty)$. Now, we choose a subsequence of $\{u_n\}_1^{\infty}$, for which the limit (2.1) is attained for $y=y_1$. From this subsequence, we choose another one, for which (2.1) is attained for $y = y_2$, etc. Then, by a diagonal operation we choose a subsequence for which we keep the same notation $\{u_n\}_1^\infty$, and over which

$$g(2y_k) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \left| u_n(x+iy_k) - u(x+iy_k) \right|^p dx = 0$$

for all $k \ge 1$. Then, in virtue of remark 1.1, for any $n \ge 1$ and $\rho > 0$ the function $u_n(z+i\rho)$ belongs to h^p. Note that in particular this is so for $\rho = y_k$ $(k = 1, 2, ...)$. By (2.2), for any fixed $k \ge 1$ the sequence $\{u_n(z+iy_k)\}_{n=1}^\infty$ is fundamental in h^p, and consenquently $u_n(z+iy_k) \to U(z+iy_k) \in h^p$ as $n \to \infty$ in the norm of h^p over G^+. Hence, u_n uniformly tends to U inside G^+, and $U \in h^p$ in any half-plane G_ρ^+. Thus, we conclude that (1.1) is true for U and, in addition, for any number $A > 0$

$$\iint\limits_{\substack{|x|<A \\ 1/A<y<A}} \left| U(z) - u(z) \right|^p d\mu_\omega(z) \le 2^{n-1} \left\{ \iint\limits_{\substack{|x|<A \\ 1/A<y<A}} \left| U(z) - u(z) \right|^p d\mu_\omega(z) + \right.$$

$$\left. + \iint\limits_{\substack{|x|<A \\ 1/A<y<A}} \left| U(z) - u(z) \right|^p d\mu_\omega(z) \right\} \to 0 \text{ as } n \to \infty .$$

The passage $A \to +\infty$ gives $\|U - u\|_{L_\omega^p} = 0$.

Now, let us prove a theorem on an explict form of the orthogonal projection of the space L_ω^2 to its harmonic subspace h_ω^2. Assuming that $\omega \in \tilde{\Omega}_\alpha$, $\alpha > -1$, we shall deal with the Cauchy-type kernel

$$G_\omega(z) := \int_0^{+\infty} e^{itz} \frac{dt}{I_\omega(t)}, \quad I_\omega(z) := \int_0^{+\infty} e^{-tx} d\omega(x),$$

which is holomorphic function in G^+ [8]. Note that by Lemma 3.1 of [8] for any $\omega \in \tilde{\Omega}_\alpha$ with $\alpha > -1$, any numbers $\alpha > -1$, $\rho > 0$ and any noninteger $\beta \in ([\alpha]-1, \alpha)$ there exists a constant $M_{\rho,\beta} > 0$ such that

$$\left| C_\omega(z) \right| \le \frac{M_{\rho,\beta}}{|z|^{2+\beta}}, \quad z \in G_\rho^+ := \{z : \operatorname{Im} z > \rho\}$$

Under the same assumption, we use the Green type potentials by means of the elementary Blaschke type factor

$$b_\omega(z,\zeta) := \exp\left\{ \int_0^{2\operatorname{Im}\zeta} C_\omega(z-\zeta+it)\,\omega(t)\,dt \right\}, \ \operatorname{Im} z > \operatorname{Im}\zeta > 0$$

(see formula (23) in [5]), which is a holomorphic function in G^+, where it has a unique, simple zero at $z = \zeta$.

Theorem 2.1. *If* $\omega \in \tilde{\Omega}_\alpha \left(-1 < \alpha < +\infty\right)$, *then the orthogonal projection of* L^2_ω *to* h^2_ω *can be written in the form*

$$P_\omega u(z) = \frac{1}{\pi} \iint_{G^+} u(w) \operatorname{Re}\left\{C_\omega\left(z - \overline{w}\right)\right\} d\mu_\omega(w), \; z \in G^+. \tag{2.4}$$

Proof. Let $u \in L^2_\omega$. Then, applying the estimate (2.3), where $\beta = \alpha - \varepsilon$ with a small $\varepsilon > 0$, and Holder's inequality, one can be convinced that the integral of (2.4) is absolutely and uniformly convergent inside G^+, and hence it represents a harmonic function there. Besides, using that estimate (2.3) and Holder's inequality one can prove that for any fixed $\rho > 0$ and $\varepsilon > 0$ small enough there exists a constant $M'_{\rho,\varepsilon} > 0$ depending only on ρ and ε, such that $\left|P_\omega u\left(\mathrm{Re}^{i\vartheta}\right)\right|^2 \le M'_{\rho,\varepsilon} R^{-(3+2\alpha-2\varepsilon)}$ $\left(\arcsin\frac{\rho}{R} < \vartheta < \pi - \arcsin\frac{\rho}{R}\right)$ for $R > 0$. Hence, $P_\omega u$ satisfies (1.1). Thus, it remains to show that P_ω is a bounded operator which maps L^2_ω to h^2_ω and is identical on h^2_ω.

If $u \in L^2_\omega$, then for fixed $z = x + iy \in G^+$ and $\zeta = \xi + i\eta$

$$\begin{aligned} P_\omega u(z) &= \operatorname{Re}\left\{\frac{1}{\pi}\int_0^{+\infty}\left(\lim_{R\to+\infty}\int_R^{-R} u(\zeta)\,d\xi \int_0^{+\infty} e^{it(z-\bar{\zeta})}\frac{dt}{I_\omega(t)}\right)d\omega(2\eta)\right\} = \\ &= \operatorname{Re}\left\{\frac{1}{\pi}\int_0^{+\infty}\left(\lim_{R\to+\infty}\int_0^{+\infty} e^{itz}\frac{e^{-t\eta}}{I_\omega(t)}\,dt\int_{-R}^{R} e^{-t\xi} u(\zeta)\,d\xi\right)d\omega(2\eta)\right\} = \\ &= \operatorname{Re}\left\{\frac{1}{\sqrt{\pi}}\int_0^{+\infty} d\omega(2\eta)\int_0^{+\infty} e^{itz}\frac{e^{t\eta}}{I_\omega(t)}\hat{u}_\eta(t)\right\}dt, \end{aligned} \tag{2.5}$$

where $\hat{u}_\eta(t) = 1.i.m.R \to +\infty \int_{-R}^{R} e^{it\xi} u(\xi + i\eta)\,d\xi$ is the Fourier transform of $u(\xi + i\eta) \in L^2(-\infty, +\infty)$ for a fixed, almost every $\eta > 0$. Note that the equalities in (2.5) are true, since by Plancherel's theorem

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-t\eta}}{I_\omega(t)}\left|\frac{1}{\sqrt{\pi}}\int_{-R}^{R} e^{-it\xi}u(\zeta)\,d\xi - \hat{u}_\eta(t)\right|dt$$

$$\le \left[C_{\tilde{\omega}}\left(2i(y+v)\right)\right]^{1/2}\left\|\frac{1}{\sqrt{\pi}}\int_{-R}^{R} e^{-it\xi}u(\zeta)\,d\xi - \hat{u}_\eta(t)\right\|_{L^2(-\infty,+\infty)} \to 0$$

as $R \to +\infty$. From (2.5) we conclude that

$$P_\omega u(z) = \operatorname{Re}\left\{\frac{1}{\sqrt{\pi}}\int_0^{+\infty} e^{itz}\frac{\Phi(t)}{\sqrt{I_\omega(t)}}\,dt\right\}, \; z \in G^+, ,$$

where

$$\Phi(t) := \frac{1}{\sqrt{I_\omega(t)}}\int_0^{+\infty} e^{-t\eta}\hat{u}_\eta(t)\,d\omega(2\eta).$$

The change of the integration order transforming (2.5) to (2.6) is valid, since by (2.3) for a fixed $y>0$ and a small $\varepsilon>0$ there is a constant $M>0$ such that

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty} d\omega(2\eta)\int_0^{+\infty}\frac{e^{-t(y+\eta)}}{I_\omega(t)}\left|\hat{u}_\eta(t)\right|dt \le$$

$$\le \sqrt{2}\int_0^{+\infty}\left[C_{\tilde{\omega}}\left(2i\left(y+\eta\right)\right)\right]^{1/2}\left\|\hat{u}_\eta\right\|_{L^2(0,+\infty)} d\omega(2\eta)$$

$$\le M\sqrt{2}\left\|u\right\|_{L^2_\omega}\left(\int_0^{+\infty}\frac{d\omega(2\eta)}{\left(y+\eta\right)^{3+2\alpha-\varepsilon}}\right)^{1/2} < +\infty ,$$

where $\tilde{\omega}$ is the Volterra square of ω (see lemma 4 in [5]). By an application of Holder's inequality and Plancherel's theorem, from (2.7) we get $\left\|\Phi\right\|_{L^2(0,+\infty)} \le \sqrt{2}\left\|u\right\|_{L^2_\omega}$, while by the Paley-Winener theorem (see eg. [6], pp. 130-131) from (2.7) we obtain

$$\left\|P_\omega u\right\|^2_{L^2_{\tilde{\omega}}} \le \frac{1}{\pi}\int_0^{+\infty} d\omega(2y)\int_0^{+\infty} e^{-2yt}\frac{\left|\Phi(t)\right|^2}{I_\omega(t)}dt = 2\left\|\Phi\right\|^2_{L^2(0,+\infty)} .$$

Thus, P_ω is a bounded operator which maps L^2_ω to h^2_ω.

Now, let $u \in h^2_\omega$. Then obviously $u(z+i\eta)\in h^2$ for any $\eta>0$. Hence, for any fixed $\eta>0$ the function $u(z+i\eta)$ is the real part of some function $f(z+i\eta)$ from the holomorphic Hardy space H^2 in G^+. Consequently, by the Paley-Wiener theorem

$$f(z+i\eta) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{itz}\hat{f}_\eta(t)\,dt, z\in G^+ ,$$

where

$$\hat{f}_\eta(t) = \underset{R\to+\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} e^{-it\xi} f(\xi+i\eta)\,d\xi$$

is the Fourier transform of f on the level $i\eta$, and

$$\left\|f(z+i\eta)\right\|^2_{L^2(-\infty,+\infty)} = \left\|f(z+i\eta)\right\|^2_{H^2} = \left\|\hat{f}_\eta\right\|^2_{L^2(0,+\infty)} .$$

Note that one can prove the independence of the function $e^{t\eta}\hat{f}_\eta(t)$ of $\eta>0$. Further, for any $\eta>0$ and $\zeta=\xi+i\eta$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(\xi+i\eta)C_\omega\left(z-\overline{\zeta}\right)d\xi = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{i(z+i\eta)t}\hat{u}_\eta(t)\frac{dt}{tI_\omega(t)}$$

$$= \frac{1}{2\sqrt{2\pi}}\int_0^{+\infty} e^{i(z+i\eta)t}\left[\hat{f}_\eta(t)+\hat{\overline{f}}_\eta(t)\right]\frac{dt}{tI_\omega(t)} .$$

From (2.8) and the Paley-Wiener theorem, it follows that for $t>0$

$$0 = \overline{\hat{f}_\eta(-t)} = \lim_{R\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} e^{-it\xi}\,\overline{f(\xi+i\eta)\,d\xi} = \hat{\overline{f}}_\eta(t) .$$

Consequently, for any $z\in G^+_\eta$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} u(\xi+i\eta)\,C_\omega\left(z-\overline{\zeta}\right)d\xi=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{i(z+i\eta)t}\hat{f}_\eta(t)\frac{dt}{tI_\omega(t)},$$

and hence

$$P_\omega u(z)=\operatorname{Re}\left\{\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{izt}\frac{dt}{tI_\omega(t)}\int_0^{+\infty} e^{-2t\eta}\left\{e^{t\eta}\hat{f}_\eta(t)\right\}d\omega(2\eta)\right\}=$$

$$=\operatorname{Re}\left\{\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{i(z-i\eta)t}\hat{f}_\eta(t)\,dt\right\}=\operatorname{Re}\left\{f(z)\right\}=u(z),$$

i.e. the operator P_ω is identical on h_ω^2.

3. Orthogonal decomposition. In virtue of Remark 2 and Theorem 2 in [5], if $\omega\in\tilde{\Omega}_\alpha\,(\alpha>-1)$ and υ is the associated Riesz measure of a subharmonic in G^+ function $U\in L_\omega^1$ satisfying (1.1) with $p=2$, then

$$\iint_{G^+}\left(\int_0^{2\operatorname{Im}}\omega(t)\,dt\right)d\upsilon(\zeta)<+\infty \quad\text{and}\quad \iint_{G^+}\operatorname{Im}\zeta\, d\upsilon(\zeta)<+\infty$$

for any $\rho>0$, which conditions provide the convergence of the potential

$$P_\omega(z)=\iint_{G^+}\log\left|b_\omega(z,\zeta)\right|d\upsilon(\zeta)$$

in G^+, and U is representable in the form

$$U(z)=\iint_{G^+}\log\left|b_\omega(z,\zeta)\right|d\upsilon(\zeta)+\frac{1}{\pi}\iint_{G^+}U(\omega)\left\{\operatorname{Re}\mathrm{C}_\omega\left(z-\overline{w}\right)\right\}d\mu_\omega(w)$$

$$:=G_\omega(z)+u_\omega(z),\quad z\in G^+. \qquad (3.1)$$

The next theorem gives an orthogonal decomposition for some ω-weighted classes of functions subharmonic in G^+.

Theorem 3.1. *If* $\omega\in\tilde{\Omega}_\alpha$ *with* $-1<\alpha<+\infty$*, then:*

1. *Both summands* G_ω *and* u_ω *in the right-hand side of the representation (3.1) of any function* $U\in L_\omega^2\cap L_\omega^1$ *satisfying (1.1) with* $p=2$ *are of* L_ω^2.
2. *The operator* P_ω *is identical on* h_ω^2 *and it maps all Green type potentials* $G_\omega\in L_\omega^1$ *satisfying (1.1) with* $p=2$ *to identical zero.*
3. *Any harmonic function* $u\in h_\omega^2$ *is orthogonal in* L_ω^2 *to any Green type potentials* $U\in L_\omega^1\cap L_\omega^2$ *satisfying (1.1) with* $p=2$.

Proof. Let $U\in L_\omega^1\cap L_\omega^2$ be a subharmonic in G^+ function satisfying (1.1) with $p=2$. Then U is representable in the form (3.1), where $u\in h_\omega^2$ by Theorem 2.1. Hence, also $G_\omega\in L_\omega^2$ and satisfies (1.1) with $p=2$. Further, if $G_\omega\in L_\omega^1$ and satisfies (1.1) with $p=2$, then applying the operator P_ω to both sides of equality (3.1) written for G_ω we get $P_\omega G_\omega(z)\equiv 0,\ z\in G^+$. Since P_ω is the orthogonal projection of L_ω^2 to its harmonic subspace h_ω^2, we conclude that

$$\left(P_\omega U,\,G_\omega\right)_\omega=\left(P_\omega u,\,G_\omega\right)_\omega=\left(P_\omega^* u,\,G_\omega\right)_\omega=\left(u,P_\omega G_\omega\right)_\omega=0.$$

At last, if u is a function of h^2_ω and a Green type potential $G_\omega \in L^1 \cap L^2$ and satisfies (1.1) with $p=2$, then by Theorem 2.1

$$\left(u,\, G_\omega\right)_\omega = \left(P_\omega u,\, G_\omega\right)_\omega = \left(P^*_\omega u,\, G_\omega\right)_\omega = \left(u, P_\omega G_\omega\right)_\omega = 0\,.$$

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Orthogonal Decomposition in Omega-Weighted Classes of Functions Subharmonic in the Half-Plane

The paper gives a harmonic, ω-weighted, half-plane analog of W. Wirtinger's projection theorem and its $(1-r)^\alpha$-weighted extension by M. Djrbashian, also an orthogonal decomposition is obtained for some ω-weighted classes of functions subharmonic in half plane.

Ա. Մ. Ջրբաշյան, Դ. Վարգաս

Օրթոգոնալ վերլուծում կիսահարթությունում սուբհարմոնիկ ֆունկցիաների օմեգա-կշռային դասերում

Տրված է Վ. Վիրտինգերի պրոեկցիոն թեորեմի ու դրա համար Մ. Մ. Ջրբաշյանի գտած $(1-r)^\alpha$-կշռային ընդլայնման հարմոնիկ համանմանը կիսահարթության մեջ, ինչպես նաև օրթոգոնալ վերլուծություն կիսահարթությունում սուբհարմոնիկ որոշ ω-կշռային դասերում:

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Ортогональное разложение в омега-весовых классах функций, субгармонических в полуплоскости

Дан гармонический, ω-весовой, полуплоскостной аналог проекционной теоремы В. Виртингера и ее $(1-r)^\alpha$-весового расширения, найденного М. М. Джрбашяном. Установлено также ортогональное разложение в некоторых ω-весовых классах функций, субгармонических в полуплоскости.

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