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On Ramanujan Method of Solution of Equations with Analytic Functions

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Introduction and preliminaries. Let f be analytic in a circle $D(0,R)=\{z:|z|<R\}$, $f(z)=\sum_{n=1}^{\infty}a_n z^n$ and $z_0(|z_0|<R)$ be a simple root of equation

$$f(z)=1. \tag{1}$$

It is assumed that all other roots of equation (1) have moduli strictly greater than $|z_0|$. In this case the function $g(z)=\dfrac{1}{1-f(z)}$ will be analytic at least in some domain $D(0,|z_0|+\varepsilon)\setminus\{z_0\}$ ($0<\varepsilon$) and z_0 will be a simple pole of g. Let $g(z)=\sum_{n=1}^{\infty}c_n z^{n-1}$ be the power series expansion of g. It is easy to see that $c_1=1$ and

$$c_n=\sum_{k=1}^{n-1}a_k c_{n-k}\ . \tag{2}$$

According to ([1], p. 42) “Ramanujan's discourse is characteristically brief; he gives (2) and claims, with no hypotheses, that $\dfrac{c_{n-1}}{c_n}$ approaches a root of (1)”.

Let

$$\lim_{n\to\infty}\frac{c_{n-1}}{c_n}=s. \tag{3}$$

Formulas (2) and (3) form the base of the Ramanujan method [4].

The following assertions should be proved.

1. All the coefficients $\{c_n\}\neq 0$ (at least starting from some positive integer).
2. The sequence $\left\{\frac{c_{n-1}}{c_n}\right\}$ converges.
3. The limit (3) is a root of (1).

Some progress in justification of this method is achieved in [1].

By Fabry's theorem [2] condition (3) implies that s is a singular point of the function g.

As in the considered above case the only singularity of g in the circle $D(0,|z_0|+\varepsilon)$ is the simple pole at z_0, we get $z_0=s$.

Remark. As shows the example of dilogarithm function $Li_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}$ the point

$$\lim_{n\to\infty}\frac{n^2}{(n-1)^2}=1$$

is not a pole.

More convincing (and independent of Fabry's theorem) justification of Ramanujan method (for an isolated pole of arbitrary order) may be obtained, using a slight generalization of the following result ([5], Ch.2, Ex.14).

Assertion. Suppose that g is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Then

$$\lim_{n\to\infty}\frac{c_{n-1}}{c_n}=z_0.$$

We intend to address in the next section all three challenges above in more general situation and supply exhaustive clarifications.

Auxiliary results. Let u and v be analytic in some domain $\mathcal{D}\subset\mathbb{C}$ functions and $z\in\mathcal{D}$.

Definition. We say that the function u is stronger at z than v (noted as $u\succ_z v$) if

$$\lim_{n\to\infty}\frac{v^{(n)}(z)}{u^{(n)}(z)}=0. \tag{4}$$

Lemma 1. *Let* $a,b\in\mathbb{C}; A,B\in\mathbb{C}\setminus\{0\}; m,p\in\mathbb{N}$ *and*

$$u(z)=\frac{A}{(a-z)^m},\ v(z)=\frac{B}{(b-z)^p}.$$

Then
1. If $|z-a|=|z-b|$ and $m>p$, then $u \succ_z v$ for any $z \neq a, z \neq b$.
2. If $0<|z-a|<|z-b|$, then $u \succ_z v$.

Proof. 1. We have

$$u^{(n)}(z)=A\frac{\Gamma(m+n)}{\Gamma(m)(a-z)^{m+n}}$$

and

$$\frac{v^{(n)}(z)}{u^{(n)}(z)}=\frac{B}{A}\cdot\frac{\Gamma(m)}{\Gamma(p)}\cdot\frac{(a-z)^m}{(b-z)^p}\cdot\left(\frac{a-z}{b-z}\right)^n\cdot\frac{\Gamma(n+p)}{\Gamma(n+m)}.$$

All the terms of this product are bounded. According to Stirling's formula

$$\lim_{n\to\infty}\frac{\Gamma(n+p)}{\Gamma(n+m)}=0$$

implying

$$\lim_{n\to\infty}\frac{v^{(n)}(z)}{u^{(n)}(z)}=0.$$

2. By the same way

$$\lim_{n\to\infty}\frac{v^{(n)}(z)}{u^{(n)}(z)}=\frac{B}{A}\cdot\frac{\Gamma(m)}{\Gamma(p)}\cdot\frac{(a-z)^m}{(b-z)^p}\cdot\lim_{n\to\infty}\frac{\Gamma(n+p)}{\Gamma(n+m)}\cdot\left(\frac{a-z}{b-z}\right)^n=0.$$

Lemma 2. *For function* u *from above*

$$\lim_{n\to\infty} n\frac{u^{(n-1)}(z)}{u^{(n)}(z)}=a-z.$$

Proof. We have

$$\lim_{n\to\infty} n\frac{u^{(n-1)}(z)}{u^{(n)}(z)}=\lim_{n\to\infty} n\frac{\Gamma(m+n-1)}{\Gamma(m)(a-z)^{m+n-1}}\cdot\frac{\Gamma(m)(a-z)^{m+n}}{\Gamma(m+n)}=a-z.$$

Note that if $m=1$ the fraction $n\dfrac{u^{(n-1)}(z)}{u^{(n)}(z)}$ is equal to $a-z$ for any n.

Proposition 3. *Let* h *be a function analytic in a domain* $\mathcal{D}$, $z\in\mathcal{D}$. *Denote by* $D(z,r)$ *the circle of convergence of the power series expansion of* h *centered at* z *and* $a\in D(z,r)$. *Let* $u(t)=\dfrac{1}{a-t}$. *Then* $u \succ_z h$.

Proof. For any $t\in D(z,r)$ we have

$$h(t)=\sum_{n=0}^{\infty}\frac{h^{(n)}(z)}{n!}(t-z)^n.$$

As a lies in the circle of convergence of h, the general term of its Taylor series expansion tends to zero

$$\lim_{n\to\infty}\frac{h^{(n)}(z)}{n!}(a-z)^n=0\,.$$

On the other hand

$$\frac{h^{(n)}(z)}{u^{(n)}(z)}=\frac{h^{(n)}(z)}{n!}(a-z)^{n+1}$$

and

$$\lim_{n\to\infty}\frac{h^{(n)}(z)}{u^{(n)}(z)}=0.$$

Proposition 4. *Let* $\{f_n\}_0^N$ *be a set of comparable (in the sense of the above definition) functions,* f_0 *be the unique strongest among them and*

$$F=\sum_{n=0}^{N} f_n.$$

Then

$$\lim_{n\to\infty} n\frac{F^{(n-1)}(z)}{F^{(n)}(z)}=\lim_{n\to\infty} n\frac{f_0^{(n-1)}(z)}{f_0^{(n)}(z)}.$$

Proof. Equality

$$\lim_{n\to\infty}\frac{F^{(n)}(z)}{f_0^{(n)}(z)}=\lim_{n\to\infty}\left(1+\sum_{k=1}^{N}\frac{f_k^{(n)}(z)}{f_0^{(n)}(z)}\right)=1$$

implies $F^{(n)}(z)\neq 0$ at least starting from some positive integer.
Further

$$\lim_{n\to\infty} n\frac{F^{(n-1)}}{F^{(n)}}=\lim_{n\to\infty}\frac{F^{(n-1)}}{f_0^{(n-1)}}\cdot n\frac{f_0^{(n-1)}}{f_0^{(n)}}\cdot\frac{f_0^{(n)}}{F^{(n)}}=\lim_{n\to\infty} n\frac{f_0^{(n-1)}}{f_0^{(n)}}.$$

Generalization of Ramanujan method. The equation to be solved is now

$$f(z)=0\,, \tag{5}$$

where f is an analytic in some domain $\mathcal{D}$ function, having a set of (not obligatory simple) zeros.

Let $z\in\mathcal{D}, f(z)\neq 0$. Denote

$$P_0(z)=1/f(z) \text{ and } P_n(z)=\left(d^n\frac{1}{f}\right)(z), n\in\mathbb{N}, \tag{6}$$

where d is the differentiation operator.

Remark ([3], formula (2.2).) If P_n is expensive to calculate, one may use the recurrences

$$P_n(z)=-\frac{1}{f(z)}\sum_{k=0}^{n-1}C_n^k P_k(z)\,f^{(n-k)}(z), n\geq 1. \tag{7}$$

Theorem. *Let* P_0 *be meromorphic in* $\mathcal{D}$ *,* $z\in\mathcal{D}, f(z)\neq 0$*. The formula*

$$a=z+\lim_{n\to\infty} n\frac{P_{n-1}(z)}{P_n(z)} \tag{8}$$

defines the nearest to z zero of f. If there are many such zeros, then a is the zero of the highest order. The limit does not exist if there are many concurrent zeros of the highest order in the same distance from z.

Proof. Let $\{a_k\}$ be nearest to z set of zeros of f. Subtracting the principal parts of the Laurent series of P_0 we get a function

$$h(z) = P_0(z) - \sum_{j,k} \frac{A_{jk}}{(a_k - z)^{m_{jk}}},$$

analytic in a circle $D(z,r)$. The proof may be completed, recalling Lemmas 1 and 2, Propositions 3 and 4.

Denote by E the set of mediatrices $E = \{z : |z - z_k| = |z - z_m|\}$, where z_k, z_m are all pairs of different zeros of f. According to section 2, the limit in (8) exists at least for any $z \in \mathbb{C} \setminus E$, i.e. almost everywhere.

So formulas (6) (or (7)) and (8) describe the generalized Ramanujan algorithm of solution of (5).

In [3] the author says "In the present study, we give analytic proof of his (Ramanujan's) method and generalize this to approximate a root of nonlinear equation with the arbitrary order of convergence" and supposes that $f(\alpha) = 0, |z - \alpha| < 1, f'(z) \neq 0.$

Let

$$H_n(z) = z - \alpha + n\frac{P_{n-1}(z)}{P_n(z)}.$$

The author writes "Since $|z - \alpha| < 1$ and letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} H_n(z) = 0."$$

The next examples show that the last assertion is erroneous.

Example 1. Let $f(z) = 6z^2 - z - 1.$ Then $\alpha_1 = -1/3, \alpha_2 = 1/2$ and

$$\frac{1}{6z^2 - z - 1} = \frac{1}{5}\cdot\frac{1}{z - 1/2} - \frac{1}{5}\cdot\frac{1}{z + 1/3}.$$

We have

$$\frac{nP_{n-1}(z)}{P_n(z)} = -\frac{\dfrac{n!}{(z-1/2)^n} - \dfrac{n!}{(z+1/3)^n}}{\dfrac{n!}{(z-1/2)^{n+1}} - \dfrac{n!}{(z+1/3)^{n+1}}}.$$

For $z_1 = 0$

$$\alpha_1 = \lim_{n\to\infty} \frac{nP_{n-1}(0)}{P_n(0)} = -\frac{1}{3}.$$

Choosing $z_2 = 1/4$ we get

$$\alpha_2 = \frac{1}{4} + \lim_{n\to\infty} \frac{nP_{n-1}(1/4)}{P_n(1/4)} = \frac{1}{2}.$$

Note that for both choices $|z_2 - \alpha_1| = 7/12 < 1$ and $|z_1 - \alpha_2| = 1/2 < 1.$

It is easy to see that the attraction basin of the root α_1 is the left half-plane $\Pi_- = \left\{ z : \operatorname{Re} z < \frac{1}{12} \right\}$ and for α_2 - the right half-plane $\Pi_+ = \left\{ z : \operatorname{Re} z > \frac{1}{12} \right\}$.

Example 2. In the figure below the attraction basins of four roots of equation $z^4 - z = 0$ are plotted. For $z_0 = 0$ the attraction basin is the inner triangle, marked by the letter Z. For roots $z_{k+1} = \exp(i2\pi k/3), k = 0,1,2$ the corresponding domains (marked by letters A, B, C respectively) are bounded by two rays and a side of the triangle.

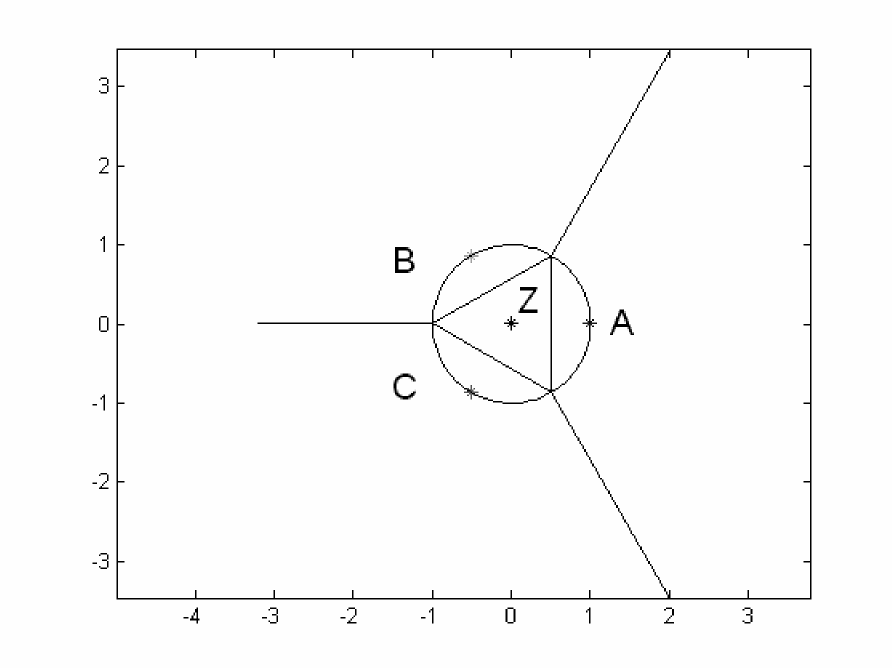

Fig. 1. Attraction basins for the roots of equation $z^4 - z = 0$.

Remark. If α is a simple root of equation (5) then the error of iterations (8) tends to zero as a geometric progression, otherwise the n-th error is of the magnitude $1/n$.

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On Ramanujan Method of Solution of Equations with Analytic Functions

We recall the Ramanujan method of solution of equations with analytic functions and mention some results which may be used for justification of it. Some auxiliary notions are introduced and pertinent propositions are proved. Then the general case is treated. A formula is proposed, permitting to find the nearest to given starting point zero of the function. If there are many such zeros, then the zero of the highest order is found and the limit does not exist if there are many concurrent zeros of the highest order in the same distance from the starting point. A description of admissible starting values of iterations is supplied.

Լ. Զ. Գևորգյան

Անալիտիկ ֆունկցիաներով հավասարումների լուծման Ռամանուջանի եղանակի մասին

Հիշեցվում է անալիտիկ ֆունկցիաներով հավասարումների լուծման Ռամանուջանի եղանակը, և ներկայացվում են որոշ արդյունքներ, որոնք կարող են օգտագործվել դրա հիմնավորման համար: Ներմուծվում են նոր հասկացություններ, և ապացուցվում են յուրահատուկ պնդումներ: Քննարկվում է Ռամանուջանի եղանակը ամենաընդհանուր դրվածքով: Առաջարկվում է հաշվարկային բանաձև, որը հնարավորություն է տալիս գտնելու ֆունկցիայի՝ նախապես ընտրված կետին ամենամոտ զրոն: Եթե դրանք մի քանիսն են, ապա գտնվում է ամենաբարձր կարգի զրոն, և սահմանը գոյություն չունի, եթե ամենաբարձր կարգի և միննույն հեռավորության վրա գտնվող զրոները մի քանիսն են: Նկարագրվում են նաև իտերացիաների թույլատրելի սկզբնական մոտավորությունները:

Л. З. Геворгян

О методе Рамануджана решения уравнений с аналитическими функциями

Приводятся метод Рамануджана решения уравнений с аналитическими функциями и некоторые результаты, которые могут быть использованы для его обоснования. Вводятся некоторые понятия и доказываются специальные утверждения. Обсуждается метод Рамануджана в наиболее общей постановке. Приводится расчетная формула, позволяющая найти корень фуннкции, наболее близкий к заранее выбранной точке. Если ближайших точек несколько, то находится корень наивысшей кратности, а если имеется несколько ближайших корней наивысшей кратности, то предела не существует. Описаны также допустимые начальные значения итераций.

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