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Discrete Capacity and Higher-order Differences of Two-state Markov Chains

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1. Introduction

In this paper an application of the suggested in [1]-[7] difference analysis to studying the two-state Markov chains is presented. The difference analysis is a method for studying irregular and random time series, based on consideration of higher-order absolute differences taken from the series' progressive terms. This method allowed us to reveal some new aspects in dynamical systems: *e.g.*, higher-order-difference version for Lyapunov exponent [3] and bistability of higher-order differences taken from periodic time series, have been established [6].

We study time-homogeneous Markov chains $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$, whose state space $X = \{x\}$ consists of two different items; more precisely, we suppose that $X = \{0, 1\}$, that is, each component ξ_n of $\boldsymbol{\xi}$ (which describes the chain at the moment n) is a random binary variable.

The main result of this paper, Theorem 1, is a limiting theorem for such chains: it asserts the existence of limit of kth order absolute differences taken from progressive terms of a given series $(\xi_n)_{n=0}^{\infty}$, when k converges to ∞ remaining on "large" subsets $E \subseteq \mathbb{N}$ of natural series $\mathbb{N}$. The "size" of such sets E is described in terms of some discrete capacity: such sets E are *thick* sets, defined by means of Wiener criterion type relation from potential theory (see, *e.g.*, [8] and [9]). The limiting process, whose existence asserts Theorem 1, is the equi-distributed random sequence.

The paper consists of three sections. The next Section 2 describes the statement of the considered problem, in Section 3 we present the definitions of discrete capacity, *thin* and *thick* sets, and formulate our Theorem 1.

2. **Statement of the problem**

Let us explain the statement of the problem which we study. Let

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_n, \ldots)$$

be some random sequence whose components ξ_n take binary values x from $X = \{0, 1\}$ with some positive probabilities $p_n(x)$, $P(\xi_n = x) = p_n(x)$ $(p_n(0) + p_n(1) = 1)$. Then kth order $(k \geq 0)$ absolute differences $\xi_n^{(k)}$, defined recurrently as: $\xi_n^{(0)} \equiv \xi_n$ and

$$\xi_n^{(k)} = |\xi_{n+1}^{(k-1)} - \xi_n^{(k-1)}| \qquad (n \geq 0),$$

also take binary values with some probabilities $p_n^{(k)}(x)$,

$$P(\xi_n^{(k)} = x) = p_n^{(k)}(x) \qquad (p_n^{(k)}(0) + p_n^{(k)}(1) = 1);$$

hence, one can consider kth order difference random binary sequence

$$\boldsymbol{\xi}^{(k)} = (\xi_0^{(k)}, \xi_1^{(k)}, \ldots, \xi_n^{(k)}, \ldots).$$

We are interested in existence of the limit of $\boldsymbol{\xi}^{(k)}$ when k goes to infinity. Let some infinite $\Lambda \subseteq \mathbb{N}$ be given; we say that $\boldsymbol{\xi}^{(k)}$ converge to a random binary sequence $\boldsymbol{\xi}^{(\infty)}$, if $p_n^{(k)}(x)$ $(n \in \mathbb{N}, x \in X)$ tend to some numbers $p_n^{(\infty)}(x)$ $(p_n^{(\infty)}(0) + p_n^{(\infty)}(1) = 1)$ as $k \to \infty$ and $k \in \Lambda$ (convergence by probability on Λ). Given Λ the limiting process

$$\boldsymbol{\xi}^{(\infty)} = \boldsymbol{\xi}_\Lambda^{(\infty)} = (\xi_0^{(\infty)}, \xi_1^{(\infty)}, \ldots, \xi_n^{(\infty)}, \ldots)$$

(so-called *partial limit*) is defined as random sequence, whose components $\xi_n^{(\infty)}$ take the values $x \in X$ with the probabilities $p_n^{(\infty)}(x)$.

We study time-homogeneous Markov chains $\boldsymbol{\xi}$, that is, when for x, x_i, $y \in X$

$$P(\xi_n = y | \xi_{n-1} = x, \xi_{n-2} = x_1, \ldots, \xi_0 = x_{n-1}) = P(\xi_n = y | \xi_{n-1} = x) \qquad (1)$$

(Markov property) and there is some function $\pi(x, y)$ on $X \times X$ such that

$$P(\xi_n = y | \xi_{n-1} = x) = \pi(x, y) \quad \text{for } n \geq 1 \text{ and } x, y \in X \qquad (2)$$

(homogeneity). Some computations testify, that if for such $\boldsymbol{\xi}$ an infinite $\Lambda \subseteq \mathbb{N}$ is chosen arbitrarily, the limiting process $\boldsymbol{\xi}_\Lambda^{(\infty)}$ may not exist; on the other hand, a theorem announced in [7] asserts that if $\Lambda = \{2^m - 1 : m \geq 0\}$, then $\boldsymbol{\xi}_\Lambda^{(\infty)}$ exists. The problem which studies the present paper is the following (descriptively): how "large" can be the sets $\Lambda \subseteq \mathbb{N}$ which permit the existence of $\boldsymbol{\xi}_\Lambda^{(\infty)}$, and how their "size" can be described? This paper considers the chains for which $\pi(x, y) > 0$ and

$$\pi(0,0) \neq \pi(1,1) \quad \text{and} \quad \pi(0,0) + \pi(1,1) \neq 1. \qquad (3)$$

We claim that for time-homogeneous binary Markov chains the problem stated is resolved in terms of some discrete capacity defined on $2^{\mathbb{N}}$ and corresponding *thin* (*fine*) and *thick* sets. The capacity $\mathcal{C}$, considered here, is a modification of the discrete capacity used in [4]. The solution to our problem is given by Theorem 1, which is formulated in terms of thick sets, defined by means of well-known in potential theory Wiener criterion type relation.

3. Some definitions and main theorem

We consider binary Markov chains $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$ whose state space X consists of two binary symbols, $X = \{0, 1\}$, and for which Eq. (1) holds. We assume that the chains $\boldsymbol{\xi}$ are time-homogeneous, which means that one-step transition probabilities $P(\xi_n = y | \xi_{n-1} = x)$ do not depend on time n, i.e., for some $\pi(x, y)$ Eq. (2) holds; it is also assumed that some initial distribution of probabilities $P(\xi_0 = x)$ on X is given.

To proceed to formulation of our Theorem 1, we first present the notions of discrete capacity $\mathcal{C}$ and associated with this capacity thin and thick sets. The capacity $\mathcal{C}$ is assigned on $2^{\mathbb{N}}$; to define it, we consider binary codes of natural numbers. Let $k \in \mathbb{N}$, $(k \geq 1)$ and $(\varepsilon_0, \ldots, \varepsilon_p)$ be the binary code of k: $k = \sum_{i=0}^{p} \varepsilon_i 2^i$ where $p \geq 0$, $\varepsilon_i \in \{0, 1\}$ and $\varepsilon_p = 1$ (binary expansion of k). Let $\nu(k)$ denotes the maximal of such m $(0 \leq m \leq p)$, for which all the coefficients ε_i, $0 \leq i \leq m$ of binary expansion of k are equal to 1.

Definition 1. *For $e \subseteq \mathbb{N}$ we define*

$$\mathcal{C}(e) = \sum_{k \in e} \nu(k). \tag{4}$$

A set $e \subseteq \mathbb{N}$ is called thin (or, fine) set ($\mathfrak{F}$-set) if the relation

$$\sum_{p=1}^{\infty} 2^{-p} \mathcal{C}(e \cap K_p) < \infty, \tag{5}$$

where $K_p = \{k \in \mathbb{N} : 2^p \leq k < 2^{p+1}\}$, holds. If the set $e \subseteq \mathbb{N}$ is not thin (i.e., Eq. (5) is failed), e is called thick set ($\mathfrak{T}$-set).

The $\mathcal{C}(e)$ from Eq. (4) can be expressed in terms of binomial coefficients as follows. Let (for given $k \geq 1$) $\mu(k)$ denotes the maximal of such m $(0 \leq m \leq k)$, for which all the binomial coefficients $\binom{k}{i}$, $0 \leq i \leq m$ (first m entries of kth line $(\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k})$ of the Pascale triangle), are odd numbers; one can prove that

$$\mu(k) = 2^{\nu(k)}$$

and, therefore,

$$\mathcal{C}(e) = \sum_{k \in e} \log_2 \mu(k).$$

Since for infinite collection of bounded sets $e \subset \mathbb{N}$ and some positive constant we have $\mathcal{C}(e) \leq const. \mathcal{C}(\partial e)$ (cp. [4]; such inequality is mentioned also in [10] when defining a capacity of clusters from $\mathbb{N} \times \mathbb{N}$, used in some models [11]-[12] of self-organized criticality), which is a characteristic property of classical capacities (e.g., [13]), we call $\mathcal{C}$ a capacity. We note that $\mathcal{C}$ is differed from discrete capacity, considered in denumerable Markov chains and random walk (see, e.g., [13]).

The next Proposition 1 contains some formal properties of capacity $\mathcal{C}$ and fine and thick sets (which we abbreviate as $\mathfrak{F}$-sets and $\mathfrak{T}$-sets, respectively); we note that $\mathcal{C}(e) \geq 0$ for arbitrary $e \subseteq \mathbb{N}$.

Proposition 1. *The next statements (a)-(f) are true: (a) $\mathcal{C}(\emptyset) = 0$ and $\mathcal{C}(\mathbb{N}) = \infty$. (b) If $e_1 \subseteq e_2$ then $\mathcal{C}(e_1) \leq \mathcal{C}(e_2)$. (c) $\mathcal{C}(\{2^p \leq k < 2^{p+1}\}) = (1 + o(1))2^p \quad (p \to \infty)$. (d) The $\mathbb{N}$ is $\mathfrak{T}$-set. (e) Every finite subset of $\mathbb{N}$ is $\mathfrak{F}$-set and finite union of $\mathfrak{F}$-sets is $\mathfrak{F}$-set. (f) If e is $\mathfrak{T}$-set and e' is $\mathfrak{F}$-set, then $e \cup e'$ and $e \setminus e'$ are $\mathfrak{T}$-sets.*

By using the next Proposition 2 one can construct more complicated examples of thin and thick subsets of $\mathbb{N}$.

Proposition 2. *Let for $p \geq 1$ the natural numbers $0 \leq s_p \leq p$, $s_p \to \infty$ $(p \to \infty)$ be given and $E \subseteq \mathbb{N}$ be defined as*

$$E = \bigcup_{p=1}^{\infty} \{2^p \leq k < 2^{p+1} : \nu(k) \geq s_p\}. \tag{6}$$

Then E is $\mathfrak{F}$-set if and only if for s_p the condition

$$\sum_{p=1}^{\infty} \frac{s_p}{2^{s_p}} < \infty$$

holds.

Definition 2. *A number a is called thick limit point ($\mathfrak{T}$-limit point or $\mathfrak{T}$-cluster point) of a given infinite numerical sequence a_k, $k \geq 0$ if there is a $\mathfrak{T}$-set $E \subseteq \mathbb{N}$ such that*

$$\lim_{\substack{k\to\infty \\ k\in E}} a_k = a.$$

A random binary sequence $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$ is called $\mathfrak{T}$-limit process for a given infinite series of random binary sequences $\boldsymbol{\xi}_k = (\xi_{n,k})_{n=0}^{\infty}$, $k \geq 0$ if for $x \in X$ and $n \geq 0$ the probability $P(\xi_n = x)$ is $\mathfrak{T}$-limit point for the sequence of probabilities $P(\xi_{n,k} = x)$, $k \geq 0$.

The following Theorem 1 is the main result of this paper.

Theorem 1. *Let $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$ be time-homogeneous binary Markov chain for which Eq. (3) holds. Then the equi-distributed random binary sequence is the $\mathfrak{T}$-limit process for the sequence of higher-order differences $\boldsymbol{\xi}^{(k)} = (\xi_n^{(k)})_{n=0}^{\infty}$, $k \geq 0$. More precisely, for $x \in X$ and $n \geq 0$ there is a $\mathfrak{T}$-set $E \subseteq \mathbb{N}$ of the form (6) with $\sum_{p=1}^{\infty} s_p 2^{-s_p} = \infty$, for which*

$$\lim_{\substack{k\to\infty \\ k\in E}} P(\xi_n^{(k)} = x) = \frac{1}{2}. \tag{7}$$

In certain sense, Theorem 1 can be treated as the higher-order-difference version of the classical ergodic theorem for finite (two-state) Markov chains, where some notions from potential theory are now involved.

To the end, we present some characteristics of sets E from Theorem 1 formulated in terms of their density in natural series. For $m \geq 1$ we denote $E_m = \{k \in E : 1 \leq k \leq m\}$ and consider the ratio $\rho_m(E) = \dfrac{|E_m|}{m}$ where $|E_m|$ denotes the cardinality of E_m.

Remark 1. *The sets $E \subseteq \mathbb{N}$ defined by Eq. (6) in Proposition 2 and presented in formulation of Theorem 1 are of zero density in natural series: $\rho_m(E) \to 0$ as $m \to \infty$. The sets E defined by Eq. (6) can be such that the ratio $\rho_m(E)$ converges to* 0 *as slowly as we please: given $0 < \delta_m \leq 1$, $\delta_m \downarrow 0$ the $\mathcal{T}$-set E from Theorem 1 can be constructed in such a way that $\rho_m(E) \geq \delta_m$ for all $m \geq 1$.*

In addition we note, that for chains $\boldsymbol{\xi}$ for which $\pi(0,0) = \pi(1,1)$ as well as for chains $\boldsymbol{\xi}$ for which $\pi(0,0) + \pi(1,1) = 1$ (cp. Eq. (3)) the sets E from Eq. (7) are "larger": they are thick sets with respect to capacity considered in [4] and their density equals 1.

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References

[1] *Shahverdian A. Yu., Apkarian A. V.* – Fractals. 1999. V. 7. N. 1. P. 93-103.

[2] *Shahverdian A. Yu.,* – Fractals. 2000. V. 8. N. 1. P. 49-65.

[3] *Shahverdian A. Yu., Apkarian A. V.* – Comm. Nonlin. Sci. & Comput. Simul. 2007. V. 12. P. 233-242.

[4] *Shahverdian A. Yu.* – Reports Armenian Natl. Acad. Sci. 2012. V. 112. N. 2. P. 160-169.

[5] *Shahverdian A. Yu., Kilicman A., Benosman R. B.* – Adv. Difference Equations. 2012. V. 202. P. 1-10.

[6] *Shahverdian A. Yu., Agarwal R. P., Benosman R. B.* – Adv. Difference Equations. 2014. V. 60. P. 1-9.

[7] *Shahverdian A. Yu.* – Proc. Inter. Conf. CSIT-2015. Yerevan. P. 251-252 (reprinted in: IEEE Conference Ser., CSIT-2015. P. 137-138).

[8] *Brelot M.* – Topologies and Boundaries in Potential Theory. Springer, Berlin, 1971.

[9] *Shahverdian A. Yu.* – Computational Methods and Function Theory, 2011. V. 11. N. 1. P. 71-121.

[10] *Shahverdian A. Yu.* – Reports Armenian Natl. Acad. Sci. 2011. V. 111. N. 3. P. 240-249.

[11] *Shahverdian A. Yu.* – Fractals. 1997. V. 5. N. 2. P. 199-213.

[12] *Shahverdian A. Yu., Apkarian A. V.* – Comm. Math. Sci. 2008. V. 6. N. 1. P. 217-234.

[13] *Dynkin E. B., Yushkevich A. A.* – Markov Processes: Theorems and Problems. Plenum Press, New York, 1969.

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Discrete capacity and higher-order differences of two-state Markov chains

The paper studies the time-homogeneous two-state Markov chains; the states are assumed to be binary symbols 0 and 1. The higher-order absolute differences taken from progressive states of a given chain are considered. A discrete capacity of subsets of natural series is defined and a limiting theorem for these differences, formulated in terms of Wiener criterion type relation, is presented.

Ա. Յու. Շահվերդյան

Դիսկրետ ունակություն և բինար Մարկովի շղթայից վերցված բարձր կարգի տարբերությունները

Հոդվածը վերաբերում է Մարկովի համասեռ շղթաներին, որոնց վիճակների բազմությունը կազմված է բինար 0 եվ 1 սիմվոլներից: Դիտարկվում են այդպիսի շղթայի հաջորդական վիճակներից վերցված բարձր կարգի բացարձակ տարբերությունները: Սահմանված է բնական շարքի ենթաբազմության դիսկրետ ունակություն եվ Վիների հայտանիշի տիպի տերմիններով ձևակերպված է սահմանային թեորեմ այդ տարբերությունների համար:

А. Ю. Шахвердян

Дискретная емкость и разности высшего порядка от Марковских цепей с двумя состояниями

В статье рассматриваются однородные Марковские цепи с двумя состояниями: предполагается, что это есть бинарные символы 0 и 1. Рассматриваются абсолютные разности высшего порядка, взятые от состояний цепи в последовательные моменты времени. Вводится понятие дискретной емкости подмножеств натурального ряда и в терминах типа критерия Винера формулируется предельная теорема для этих разностей.