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Chebyshev Set which Can Be Represented as a Finite Union of Convex Sets

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1. Introduction. Let C be a nonempty subset of a Banach space $(X,\|\|)$. The *metric projection* (or *set of nearest points*) of x onto C is defined by: $P_C(x)=\{y\in C:\|x-y\|=d_C(x)\}$, where $d_C(\cdot)$ is the *distance function*, i.e., $d_C(x)=\inf\{\|x-y\|:y\in C\}$. We say that C is *Chebyshev* if $P_C(x)$ is a singleton for all $x\in X$. It is easy to see that Chebyshev sets are strongly closed. Blunt [3] showed that in Euclidean finite dimensional spaces Chebyshev sets are convex. One of the most famous unsolved problems in approximation theory is: whether in a smooth reflexive Banach space (or even in a Hilbert space) every Chebyshev set is convex? Although this problem is open (see [2]), several sufficient conditions for the Chebyshev set to be convex have been obtained, until now. Here is the first important result:

Theorem 1 *(Vlasov [10]). Let X be a Banach space with rotund dual. Then any Chebyshev subset of X with continuous metric projection is convex.*

This theorem was previously obtained by Asplund [1] in Hilbert spaces. Another important result is following:

Theorem 2 *(see[4], p.193). Let X be a reflexive Banach space with the Kadec-Klee property. Then any weakly closed Chebyshev subset of X has continuous metric projection.*

In this paper we look at the Chebyshev sets which can be represented as a finite union of closed convex sets. We give positive answer without Kadec-Klee property. We consider also differentiability of distance function in that case.

2. Notation and Preliminaries. Let X be a normed space with a given norm $\|\cdot\|$, X^* be its topological dual and $\langle\cdot,\cdot\rangle$ be the duality pairing between X and X^*. A real-valued function f on X is *Gâteaux differentiable* at x if there is $x^*\in X^*$ such that

$$\forall h \in X,\ \lim_{t\to 0^+} t^{-1}(f(x+th)-f(x)) = \langle x^*, h\rangle.$$

If the limit in the definition of Gâteaux differentiability exists uniformly in h on the unit sphere of X, we say that f is *Fréchet differentiable* at x.

The normed space $(X, \|\cdot\|)$

(i) is *rotund or strictly convex* whenever for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ one has $\left\|\frac{x+y}{2}\right\| < 1$,

(ii) has the (sequential) *Kadec-Klee property* provided the weak convergence of a sequence of the unit sphere of the space is equivalent to the norm convergence of this sequence.

Recall that a sequence $(y_n)_n$ from C is a *minimizing sequence* for x if $\|x - y_n\| \to d_C(x)$. Recall also that the metric projection P_C is said to be continuous at $x \in X$ provided P_C is single-valued at x and $y_n \to P_C(x)$ whenever $x_n \to x$ and $y_n \in P_C(x_n)$. If X is strictly convex, then $y \in P_C(x)$ and $z \in (y, x)$ ensure $P_C(z) = \{y\}$. The set C is a *sun* if, for each point $x \in X$ and $y \in P_C(x)$, every point on the ray $y + \mathbb{R}_+(x - y)$ has y as a nearest point in C, where $\mathbb{R}_+ := [0, +\infty)$. This notion was introduced by Klee [7] and studied by Efimov, Steckin and Vlasov [6,9]. It is not difficult to see that every convex set is a sun. Indeed, let $x \in X$, $y \in P_C(x)$ and $\lambda > 0$, then for all $z \in C$

$$\|y + \lambda(x-y) - y\| = \lambda\|x-y\| \le \lambda\left\|x - \left(\frac{1}{\lambda}z + \left(1 - \frac{1}{\lambda}\right)y\right)\right\| = \|y + \lambda(x-y) - z\|$$

thus $y \in P_C(y + \lambda(x - y))$.

Theorem 3 *(Vlasov [9]). Let X be a smooth Banach space. Then every proximinal sun subset of X is convex.*

Recall that the set C is *proximinal* if for every $x \notin C$ the set $P_C(x)$ is not empty.

To end up this section, we denote by $x_n \xrightarrow[n\to\infty]{w} x$ the weak convergence of the sequence $(x_n)_n \subset X$ to $x \in X$.

3. The finite union of closed convex sets. The union of finitely many closed convex sets being weakly closed, we see in a smooth reflexive Banach space X with the Kadec-Klee property that a subset C of X is convex whenever $C = \bigcup_{i=1}^{n} C_i$ where C_i are closed convex sets. Our aim in this section is to remove for such a set C the Kadec-Klee assumption of the norm.

We start with some properties of sets which can be represented as a finite union of closed convex sets.

Proposition 1. *Let X be a normed space and let $C = \bigcup_{i=1}^{m} C_i$ be a union of finitely many closed subsets of X. Then for any $x \in X$,*

(a) $P_C(x) = \bigcup_{i\in J} P_{C_i}(x)$, where $J = \{i : 1 \le i \le m, d_C(x) = d_{C_i}(x)\}$,

(b) there is $\delta > 0$ such that for all $u \in \mathbb{B}(x,\delta)$

$$\max_{i\in J} d_{C_i}(u) < \min_{i\in J^c} d_{C_i}(u), \quad \text{where } J^c = \{1,2,...,m\} \setminus J.$$

Proof. (a) It is evident that $d_C(x) = \min_{1\le i\le m} d_{C_i}(x)$ and therefore $J \ne \varnothing$. Let $j \in J$ and $z \in P_{C_j}(x)$. By the definition of J and C we have $d_C(x) = d_{C_j}(x) = \|x - z\|$ *and* $z \in C$, which means, that $z \in P_C(x)$. Now, let $z \in P_C(x)$, then $C = \bigcup_{i=1}^{m} C_i$ and consequently $z \in C_j$ for some j, $1 \le j \le m$. We deduce that $d_C(x) = \min_{1\le i\le m} d_{C_i}(x) \le d_{C_j}(x) \le \|x - z\| = d_C(x)$, and thus $j \in J$ and $z \in P_{C_j}(x)$.

(b) By the definition of J we have that

$$\max_{i\in J} d_{C_i}(x) = d_C(x) < \min_{i\in J^c} d_{C_i}(x). \tag{1}$$

The continuity of $u \mapsto \max_{i\in J} d_{C_i}(u)$ and $u \mapsto \min_{i\in J^c} d_{C_i}(u)$ and (1) ensure the existence of $\delta > 0$ satisfying $\max_{i\in J} d_{C_i}(u) < \min_{i\in J^c} d_{C_i}(u), \ \forall u \in \mathbb{B}(x,\delta).$

Theorem 4. *Let X be a smooth reflexive Banach space. Let C be a Chebyshev subset of X with $C = \bigcup_{i=1}^{m} C_i$ where C_i are closed convex sets. Then C is convex.*

Proof. By Theorem 3 it is sufficient to show that C is sun. Let us prove the sun property of C. Suppose that $x \notin C$ and $P_C(x) = y$. Put $\sigma = \sup\{t \ge 0 : y = P_C(q_t)\}$, where $q_t = y + t(x - y)$. We want to show that $\sigma = +\infty$. Suppose that $\sigma < +\infty$. Then we have

$$d_C(q_\sigma) = \lim_{t\nearrow\sigma} d_C(y + t(x - y)) = \lim_{t\nearrow\sigma} \|y + t(x - y) - y\| = \|q_\sigma - y\|,$$

that is $y \in P_C(q_\sigma)$ and therefore $P_C(q_\sigma) = y$. Let J and J^c denote as in Proposition 1, $J = \{i : 1 \le i \le m, d_C(q_\sigma) = d_{C_i}(q_\sigma)\}$ and $J^c = \{1,2,...,m\} \setminus J$. Then, by Proposition 1, we have

$$P_C(q_\sigma) = \bigcup_{i\in J} P_{C_i}(q_\sigma) \tag{2}$$

and there is $\delta > 0$ such that for all $u \in \mathbb{B}(q_\sigma, \delta)$

$$\max_{i\in J} d_{C_i}(u) < \min_{i\in J^c} d_{C_i}(u). \tag{3}$$

By the non-vacuity of $P_{C_i}(x)$ we get from (2) that

$$P_{C_i}(q_\sigma) = y \quad \text{for all } i \in J. \tag{4}$$

Let $\sigma' > \sigma$ such that $y + \sigma'(x - y) = q_{\sigma'} \in \mathbb{B}(q_\sigma, \delta)$, (3) provides

$$d_C(q_{\sigma'}) = \min_{1\le i\le m} d_{C_i}(q_{\sigma'}) = \min_{i\in J} d_{C_i}(q_{\sigma'}). \quad (5)$$

As C_i is a convex and hence a sun, (4) ensures $d_{C_i}(q_{\sigma'}) = \|q_{\sigma'} - y\|$ for all $i\in J$. Finally we get that $d_C(q_{\sigma'}) = \min_{i\in J} d_{C_i}(q_{\sigma'}) = \|q_{\sigma'} - y\|$, or equivalently $y = P_C(q_{\sigma'})$. This contradicts the definition of σ and the proof is completed.

4. Differentiability of the distance function of finite union of convex sets. Proposition 2. *Let X be a reflexive Banach space with the Kadec-Klee property. Let $C = \bigcup_{i=1}^{m} C_i$ where C_i are weakly closed subsets of X and $x\in X\setminus C$, $P_C(x) = \{y\}$. Then P_C is continuous at x.*

Proof. Let $(y_n)_n$ be a minimizing sequence in C for x, i.e., $y_n \in C$ *and* $\|x - y_n\| \xrightarrow[n\to\infty]{} d_C(x)$. We want to show that $(y_n)_n$ converges to y. Suppose the contrary. Without loss of generality we may assume that, for some $\alpha > 0$,

$$\|y_n - y\| > \alpha, \quad \forall n\in\mathbb{N}. \quad (6)$$

By the definition of C there is an increasing sequence $(q(n))_n$ in $\mathbb{N}$ and $j\in\{1,2,...,n\}$ such that $y_{q(n)}\in C_j$ for all $n\in\mathbb{N}$. As C_j is weakly closed and $(y_{q(n)})_k$ is bounded there exists a subsequence $(l(n))_n$ of $(q(n))_n$ and $y'\in C_j$ such that $(y_{l(n)})_n$ converges weakly to y'. We obtain that

$$d_C(x) \le \|x - y'\| \le \liminf_{n\to\infty} \|x - y_{l(n)}\| \le \limsup_{n\to\infty} \|x - y_n\| = \|x - y\| = d_C(x),$$

and therefore $y' = y = P_C(x)$ and

$$x - y_{l(n)} \xrightarrow[k\to\infty]{w} x - y \text{ and } \|x - y_{l(n)}\| \xrightarrow[k\to\infty]{} \|x - y\|.$$

The Kadec-Klee property ensures that $y_{l(n)} \to y$. This is in contradiction with (6) and the proof is completed.

Theorem 5. *Let X be a strictly convex reflexive Banach space with the Kadec-Klee property. Let $C = \bigcup_{i=1}^{m} C_i$, where C_i are closed convex subsets of X and let $x\in X\setminus C$. Then for any $\varepsilon > 0$ there is an open subset $U \subset \mathbb{B}(x,\varepsilon)$ such that P_C is continuous on U.*

Proof. Let $\varepsilon > 0$. Our aim is to find $z\in X$ and $\delta > 0$ such that $\mathbb{B}(z,\delta) \subset \mathbb{B}(x,\varepsilon)$ and P_C is singleton on $\mathbb{B}(z,\delta)$ which will ensure by Proposition 2 the continuity of P_C on $int(\mathbb{B}(z,\delta))$. As C_i are convex and X strictly convex and reflexive, $P_{C_i}(u)$ are singleton for all $u\in X$ and Proposition 2 provides $P_C(u) \ne \varnothing$ for all $u\in X$. Suppose that there is $x_1 \in B\left(x, \varepsilon/3\right)$ such that $P_C(x_1)$ contains at least two different points (otherwise the proof is finished), say $y_1^1, y_1^2 \in P_C(x_1)$. By Proposition 2 there is $j_1^1, j_1^2 \in \{1,2,...,m\}$ such that $P_{C_{j_1^1}}(x_1) = y_1^1$ and $P_{C_{j_1^2}}(x_1) = y_1^2$. It is easy to see that $y_1^1 \notin C_{j_1^2}$. Indeed if $y_1^1 \in C_{j_1^2}$ then $d_C(x_1) = \|x_1 - y_1^1\| = \|x_1 - y_1^2\| = d_{C_{j_1^2}}(x_1)$ provides $y_1^1 \in P_{C_{j_1^2}}(x_1) = y_1^2$ which is a contradiction. Pick $z_1 \in \left(y_1^1, x_1\right) \cap \mathbb{B}\left(x_1, \frac{\varepsilon}{3}\right)$. The strict convexity of the norm and Proposition 2 ensure

$$P_C(z_1) = y_1^1 \notin C_{j_1^2} \text{ and } d_C(z_1) < d_{C_{j_1^2}}(z_1). \tag{7}$$

The continuity of d_C ensures the existence of $\delta_1 \in \left(0, \frac{\varepsilon}{3}\right)$ such that

$$d_C(u) = \min_{1 \le i \le m} d_{C_i}(u) < d_{C_{j_1^2}}(u) \quad \forall u \in \mathbb{B}(z_1, \delta_1).$$

Consequently, we obtain that

$$P_C(u) \in \bigcup_{\substack{1 \le i \le m, \\ i \ne j_1^2}} P_{C_i}(u) \quad \forall u \in \mathbb{B}(z_1, \delta_1) \subset \mathbb{B}(x, \varepsilon).$$

Suppose that there is $x_2 \in \mathbb{B}\left(z_1, \frac{\delta_1}{3}\right)$ such that $P_C(x_2)$ contains at least two points (otherwise the proof is finished). By the same reasoning as above we find $j_2^2 \in \{1,2,...,m\} \setminus \{j_1^2\}$, $z_2 \in \mathbb{B}\left(z_1, \frac{\delta_1}{3}\right)$ and $\delta_2 \in \left(0, \frac{\delta_1}{3}\right)$ such that

$$d_C(u) = \min_{1 \le i \le m} d_{C_i}(u) < \min\{d_{C_{j_1^2}}(u), d_{C_{j_2^2}}(u)\} \quad \forall u \in \mathbb{B}(z_2, \delta_2) \subset \mathbb{B}(z_1, \delta_1).$$

By repeating the same process, after finite steps we obtain that there exist $k \in \{1,2,...,m\}$, $z \in X$ and $\delta > 0$ such that $d_C(u) = d_{C_k}(u) < \min_{1 \le i \le m, i \ne k} d_{C_i}(u)$ $\forall u \in B(z, \delta) \subset B(x, \varepsilon)$. Thus we conclude, by Proposition1, that for all $u \in \mathbb{B}(z, \delta)$ $P_C(u) = P_{C_k}(u)$ and therefore $P_C(u)$ is a singleton on $\mathbb{B}(z, \delta)$. The proof is finished.

Theorem 6. *(Fitzpatrick [5]) Let C be a closed subset of a Banach space X with Gâteaux (respectively Fréchet) differentiable norm off zero. Let $x \notin C$. If every minimizing sequence in C for x converges, then d_C is Gâteaux (respectively Fréchet) differentiable at x.*

Theorem 7. *Assume that X is a strictly convex reflexive Banach space whose norm is Gâteaux (respectively Fréchet) differentiable off zero and has the Kadec-Klee property. Let $C = \bigcup_{i=1}^{n} C_i$ where C_i are closed convex subsets of*

X. Let $x \notin C$. Then for any $\varepsilon > 0$ there is an open subset $U \subset \mathbb{B}(x, \varepsilon)$ such that d_C is Gâteaux (respectively Fréchet) differentiable on U.

Proof. The proof is a combination of Theorem 5 and Theorem 6.

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Chebyshev Set which Can Be Represented as a Finite Union of Convex Sets

We investigate the convexity of Chebyshev sets. It is well known that in a smooth reflexive Banach space with the Kadec-Klee property every weakly closed Chebyshev subset is convex. We prove that the Kadec-Klee property is not required when the Chebyshev set is represented by a finite union of closed convex sets. We also show that, if the norm is Gâteaux (respectively Fréchet) differentiable, then the distance function from the finite union of closed convex sets is Gâteaux (respectively Fréchet) differentiable at the points of some open dense subset.

Տ. Ա. Զաքարյան

Ուռուցիկ բազմությունների վերջավոր միավորման տեսքով ներկայացվող չեբիշևյան բազմությունների մասին

Մենք հետազոտում ենք չեբիշևյան բազմությունների ուռուցիկությունը: Հայտնի է, որ Կադեց-Կլեե հատկությամբ օժտված ողորկ ռեֆլեքսիվ բանախյան տարածություններում ամեն թույլ փակ չեբիշևյան բազմությունն ուռուցիկ է: Մենք ապացուցում ենք, որ Կադեց-Կլեե հատկությունն անհրաժեշտ չէ, եթե չեբիշևյան բազմությունը փակ ուռուցիկ բազմությունների վերջավոր միավորումն է: Ապացուցվում է նաև, որ փակ ուռուցիկ բազմությունների վերջավոր միավորումից հեռավորության ֆունկցիան դիֆերենցելի է Գատոյի իմաստով (համապատասխանաբար՝ Ֆրեշեի իմաստով) որոշ բաց խիտ ենթաբազմության կետերում, եթե նորման դիֆերենցելի է Գատոյի իմաստով (համապատասխանաբար՝ Ֆրեշեի իմաստով):

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О чебышевских множествах, допускающих представление в виде объединения конечного множества выпуклых множеств

Исследована выпуклость чебышевских множеств. Хорошо известно, что в гладких рефлексивных банаховых пространствах, удовлетворяющих свойству Кадеца – Клее, каждое слабо замкнутое чебышевское подмножество является выпуклым. В настоящей статье доказано, что свойство Кадеца – Клее не является необходимым, если чебышевское множество является конечным объединением замкнутых выпуклых множеств. Также показано, что если норма дифференцируема

по Гато (соответственно по Фреше), то функция расстояния от конечного объединения замкнутых выпуклых множеств также дифференцируема по Гато (соответственно по Фреше) в точках некоторого открытого всюду плотного подмножества.

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