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Characterization of Spectraloid and Normaloid Operators

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0. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and A be a bounded linear operator, acting in H. Denote by SpA the spectrum of A. Recall that $\lambda \in$ J is said to be a reducing (normal) eigenvalue of A if there exists a nonzero element x such that $Ax = \lambda x,\ A^*x = \bar{\lambda}x$ and λ belongs to the approximate point spectrum $\pi(A)$ of A if there exists a sequence of unit vectors $\{x_n\} \subset H$ such that

$$\|(A-\lambda I)x_n\| \to 0. \tag{1}$$

We say that λ is an approximate normal eigenvalue if

$$\|(A-\lambda I)x_n\| + \|(A-\lambda I)^* x_n\| \to 0. \tag{2}$$

Denote by $W(A)$ the numerical range of A, i.e. $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$ and lcet $r(A) = \sup\limits_{\lambda \in SpA} |\lambda|$ be the spectral radius of A, $w(A) = \sup\limits_{\lambda \in W(A)} |\lambda|$ be the numerical radius of A. As it is well known [9],[10] $r(A) \le w(A)$, the numerical radius is an operator norm, equivalent to the usual one, i.e. $w(A) \le \|A\| \le 2w(A)$, so $1 \le \frac{\|A\|}{w(A)} \le 2$. The extremal values of this ratio are investigated in [3]. The operator, satisfying $r(A) = w(A)$ is said to be spectraloid and if $w(A) = \|A\|$ (equivalent condition $r(A) = \|A\|$), then A is said to be normaloid, so any normaloid operator is spectraloid.

The operator A satisfies the Daugavet equality [2] if $\|A+I\|=\|A\|+1$. We introduce a slightly more general notion and say that A satisfies the Daugavet equality at λ if

$$\|A+\lambda I\|=\|A\|+|\lambda|. \tag{3}$$

1. We start by describing an elementary necessary and sufficient condition for on operator to be spectraloid or normaloid.

Lemma 1. *A Hilbert space operator A is*

i) normaloid if and only if there exists a complex number λ such that $|\lambda|=\|A\|$ and $\lambda\in SpA$;

ii) spectraloid if and only if there exists a complex number λ such that $|\lambda|=w(A)$ and $\lambda\in SpA$.

Note that the mentioned above conditions imply that in both cases $\lambda\in\partial W(A)\cap\partial SpA$, where ∂M is the topological boundary of the set M.

Corollary. Properties of operators to be spectraloid or normaloid are translation-invariant at least in one direction.

Proof. For any $\mu\in\mathbb{C}$ one has $Sp(A+\mu I)=SpA+\mu$, $W(A+\mu I)=W(A)+$ $+\mu$ and if $\arg\lambda=\arg\mu$, then $r(A+\mu I)=r(A)+|\mu|$, $w(A+\mu I)=w(A)+$ $+|\mu|$.Thus for spectraloid operator $r(A+\mu I)=$ $w(A+\mu I)$ and for normaloid operator

$$\|A\|+|\mu|=w(A)+|\mu|=w(A+\mu I)\le\|A+\mu I\|\le\|A\|+|\mu|. \tag{4}$$

Proposition 1. The operator A is normaloid if and only if there exists a number $\lambda,|\lambda|=\|A\|$, which is an approximate normal eigenvalue of A.

Proof. The sufficiency of this condition is evident. Passing to the necessity, note that the boundary of the spectrum is contained in the approximate point spectrum ([10], Problem 63), so the condition (1) is satisfied. Denote $\lambda_n=\langle Ax_n,x_n\rangle$. From the inequality

$$|\langle Ax_n,x_n\rangle-\lambda|=|\langle(A-\lambda I)x_n,x_n\rangle|\le\|(A-\lambda I)x_n\|$$

one gets $\lambda_n\to\lambda$, hence

$$\left\|(A-\lambda I)^* x_n\right\|^2=\|A^*x_n\|^2-2\operatorname{Re}\lambda\bar{\lambda}_n+|\lambda|^2\le 2\left(\|A\|^2-\operatorname{Re}\lambda\bar{\lambda}_n\right)\to 0.$$

Note that conditions $|\lambda|=\|A\|$ and (1) imply (2).

Orland proved [14] that $\lambda\in\overline{W(A)}$ (the upper bar on the set denotes the closure of the corresponding set) and $|\lambda|=\|A\|$ imply $\lambda\in SpA$.

For the particular case the conclusion may be formulated more precisely.

Lemma 2. *Let $\lambda\in W(A)$ and $|\lambda|=\|A\|$. Then λ is a reducing eigenvalue of A.*

Proof. Let $\|A\| = |\lambda| = |\langle Ax, x\rangle|$, where $\|x\| = 1$. Then

$$\|A\| = |\langle Ax, x\rangle| \le \|Ax\|\cdot\|x\| \le \|A\|\cdot\|x\|^2 = \|A\|,$$

meaning that the Schwartz inequality becomes an equality, hence $Ax = \lambda x$. By the same way $A^* x = \bar{\lambda} x$.

This lemma is a generalization of a result by Laursen ([13], Lemma 1.10.)

Denote $D(a,r) = \{z : |z-a| \le r\}$ the circle of the radius r on the complex plane.

Example. Let S be the operator of the simple unilateral shift, realized, e.g. as the operator of the multiplication by the independent variable $(Sf)(z) = zf(z)$ in the Hardy space $H^2(D)$ in the unit circle and $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Denote by A the operator $0.5 \cdot S \oplus J_2$. Evidently $W(A) = D(0,1/2)$. Any complex number $\mu, |\mu| = 1/2$ satisfies

$$\mu \in W(A), |\mu| = w(A) = r(A),$$

but A has no normal eigenvalues. This example shows that Lemma 2, in general, is not true for spectraloid operators.

Lemma 3. *Let* $\lambda \in \pi(A), \bar{\mu} \in \pi(A^*), \lambda \ne \mu$ *and*

$$\|(A-\lambda I)x_n\| \to 0, \|(A-\mu I)^* y_n\| \to 0, \|x_n\| = \|y_n\| = 1.$$

Then $\lim_{n\to\infty} \langle x_n, y_n\rangle = 0$.

Proof. As $(\mu - \lambda)\langle x_n, y_n\rangle = \langle (A-\lambda I)x_n, y_n\rangle - \langle x_n, (A-\mu I)^* y_n\rangle$, we get

$$|\langle x_n, y_n\rangle| \le \frac{1}{|\mu-\lambda|}\left(\|(A-\lambda I)x_n\| + \|(A-\mu I)^* y_n\|\right) \to 0.$$

Proposition 2. Let $\lambda \in \partial W(A)$. Then $\lambda \in SpA$ if and only if λ is an approximate normal eigenvalue.

Proof. The sufficiency of this condition is obvious. Let now $\lambda \in \partial W(A) \cap SpA$. As the numerical range is convex, we may trace a support line l to $W(A)$, passing through λ. Let p be the perpendicular to l at λ, directed outward to $W(A)$. Take a point $\mu \in p$. As $SpA \subset \bar{W}(A)$, then $\mu \notin SpA$. According to well-known result of Stone [15] for the resolvent $R_\mu(A) = (A-\mu I)^{-1}$ the following inequality is satisfied $\|R_\mu(A)\| \le \frac{1}{dist(\mu, W(A))}$, where $dist(\mu, W(A))$ is the distance from μ to $W(A)$. By choice of μ one has $dist(\mu, W(A)) = |\lambda - \mu|$ and $\|R_\mu(A)\| \le |\lambda - \mu|^{-1}$. According to the spectral mapping theorem $\lambda \in SpA$ implies $(\lambda - \mu)^{-1} \in SpR_\mu(A)$. As for any operator its norm is not less than the spectral radius $\|R_\mu(A)\| \ge \frac{1}{dist(\mu, SpA)} = |\lambda - \mu|^{-1}$, finally $\|R_\mu(A)\| = |\lambda - \mu|^{-1}$. Recalling Proposition 1, we get

$$\left\|\left(R_\mu(A) - \frac{1}{\lambda - \mu} I\right) x_n\right\| + \left\|\left(R_\mu(A) - \frac{1}{\lambda - \mu} I\right)^* x_n\right\| \to 0.$$

As

$$A - \lambda I = (\mu - \lambda)(A - \mu I)\left(R_\mu(A) - \frac{I}{\lambda - \mu}\right)$$

and

$$(A - \lambda I)^* = (\bar{\mu} - \bar{\lambda})(A - \mu I)^* \left(R_\mu(A) - \frac{I}{\lambda - \mu}\right)^*,$$

finally we have $\|(A - \lambda I)x_n\| + \|(A - \lambda I)^* x_n\| \to 0$.

This is the infinite dimensional analogue of Theorem 1.6.6 from [11]. As simple consequence we get the following (known)

Corollary. Any eigenvalue, belonging to the boundary of the numerical range is a normal eigenvalue.

Note that conditions $|\lambda| = w(A)$ and (1) imply (2).

Combining Lemma 1 and Proposition 2, we get the following result.

Proposition 3. The operator A is spectraloid if and only if there exists a number $\lambda, |\lambda| = w(A)$ which is an approximate normal eigenvalue of A.

Lemma 4. *Let* $\lambda, \mu \in \pi(A) \cap \partial W(A), \lambda \ne \mu$ *and*

$$\|(A - \lambda I)x_n\| \to 0, \|(A - \mu I)y_n\| \to 0, \|x_n\| = \|y_n\| = 1.$$

Then $\lim_{n\to\infty} \langle x_n, y_n \rangle = 0$.

Proof. By Proposition 2 $\|(A - \mu I)y_n\| \to 0$ implies $\|(A - \mu I)^* y_n\| \to 0$. The proof is completed, recalling Lemma 3.

Lemma 5. *Let* $\lambda \in \partial W(A) \cap SpA$. *Then* λ *is either a normal eigenvalue of* A *or there exists a sequence* $\{x_n\} \subset H$ *of unit vectors, converging weakly to the neutral element and satisfying (2).*

Proof. Let $\{x_n\}$ be a sequence of unit vectors satisfying (1). As the unit sphere in the Hilbert space is sequentially weak compact there exists a subsequence (denoted again by the same letter) such that $x_n \xrightarrow{w} x$. Let ε be a positive number. Choosing a subsequence, we may assume that $\|(A-\lambda I)x_n\| < \varepsilon/2$, $n \in \mathbb{N}$. According to Mazur's theorem ([12], Chapter V, 1, Theorem 2) if $x_n \xrightarrow{w} x$, then for any ε there exist a convex combination $\sum_{n=1}^{N} \alpha_n x_n \left(\alpha_n > 0, \sum_{n=1}^{N} \alpha_n = 1\right)$ such that $\left\|x - \sum_{n=1}^{N} \alpha_n x_n\right\| < \varepsilon$. Choosing N from the condition $\left\|x - \sum_{n=1}^{N} \alpha_n x_n\right\| < \varepsilon / 2\|A-\lambda I\|$, we get

$$\|(A-\lambda I)x\| < \left\|(A-\lambda I)\left(x - \sum_{n=1}^{N} \alpha_n x_n\right)\right\| + \left\|(A-\lambda I)\sum_{n=1}^{N} \alpha_n x_n\right\| \leq$$

$$\leq \varepsilon/2 + \sum_{n=1}^{N} \alpha_n \|(A-\lambda I)x_n\| < \varepsilon/2 + \varepsilon/2 \sum_{n=1}^{N} \alpha_n = \varepsilon.$$

If $x \neq \theta$, then x is an eigenelement of A.

A similar result, attributed to Putnam and Schechter, may be found in ([5], Theorem (3.3)).

Returning to Proposition 3, note that as $\lambda \in \partial W(A)$, for the first case λ is a normal eigenvalue. Evidently, in the finite dimensional space only this case may be realized. This remark permits to give a complete description ([11], pp. 45, 60) of spectraloid and normaloid operators in finite dimensional unitary space. In the second case λ belongs ([3], Theorem (5.1), Corollary) to the essential numerical range $W_e(A)$, hence $\overline{\lambda} \in W_e(A^*)$. Finally, we arrive at the following

Proposition 4. Let $|\lambda| = w(A)$ and $\lambda \in SpA$. Then λ is either a normal eigenvalue or $\lambda \in W_e(A)$ and $\overline{\lambda} \in W_e(A^*)$.

2. We intend to give another necessary and sufficient condition for an operator to be normaloid.

Proposition 5. The operator A is normaloid if and only if it satisfies the Daugavet equality at a nonzero complex λ.

Proof. Let first $\|A+\lambda I\| = \|A\| + |\lambda|, \lambda \neq 0$. According to a theorem of Barraa and Boumazgour [1] for any two Hilbert space operators A and B conditions $\|A+B\| = \|A\| + \|B\|$ and $\|A\| \cdot \|B\| \in \overline{W(B^* A)}$ are equivalent. Taking $B = \lambda I$, we get $|\lambda| \cdot \|A\| \in \overline{\lambda} \cdot \overline{W(A)}$ or $\|A\| e^{i \arg \lambda} \in \overline{W(A)}$. This inclusion implies $w(A) = \|A\|$. The necessity follows from relations (4).

The example of the two-dimensional Jordan block J_2, for which des–pite the equality $w(J_2 + \mu I) = w(J_2) + |\mu|, \forall \mu \in$ J we have $w(J_2) = 1/2, r(J_2) = 0$ shows that the last equality is only necessary for an operator to be spectraloid.

Remark 1. It is easy to see that the equality (3) implies $\|A+\mu I\| = \|A\| + |\mu|$ for any $\mu, \arg\mu = \arg\lambda$.

Remark 2. As the numerical range in a finite dimensional space is always closed, the equality (3) in this case is equivalent to $\|A\| e^{i\arg\lambda} \in W(A)$.

According to Lemma 2 the the equality (3) in finite dimensional space may be fulfilled only in a finite number (not exceeding the dimension of the underlying space) of directions.

As $\|A\| = \sup\limits_{\|x\|=1} |\langle Ax, x\rangle|$ for any self-adjoint operator A, the Daugavet equality for self adjoint operators is satisfied on the real axis in positive or in negative (or in both) direction. In the direction of the imaginary axis the Vidav equality $\|I + itA\| = 1 + o(t), t \in \text{Ÿ}, t \to 0$ holds [16], excluding the Daugavet equality.

For the Fourier transform operator F, which is unitary in $L^2(\text{Ÿ})$ with eigenvalues $\{\pm 1, \pm i\}$ the Daugavet equality is satisfied along the coordinate axes in both directions.

Remark 3. In [4] it has been shown that $w(A) = \|A\|$ is equivalent to the equality $\max\limits_{|t|=1} \|I + tA\| = 1 + \|A\|$.

Proposition 6. Conditions $\|A + \lambda I\| = \|A\| + |\lambda|$ for any $\lambda \in \mathbb{C}$ and $\overline{W(A)} = D(0, \|A\|)$ are equivalent.

Proof. The sufficiency follows from the result of Barraa and Boumazgour. The second part may be proved, recalling [6] the equality

$$\lim_{\substack{|\lambda|\to\infty \\ \arg\lambda=\varphi}} \left(|\lambda| - \|A - \lambda I\|\right) = \inf_{\varphi\in[0;2\pi]} \operatorname{Re}\left(e^{i\varphi} A\right).$$

According to Remark 2, the condition of the last Proposition cannot be fulfilled in a finite dimensional space. For the operator of the simple unilateral shift S

$$\|S + \lambda I\| = \sup_{|z|\le 1} |z + \lambda| = 1 + |\lambda|,$$

meaning that the Daugavet equality is satisfied for any $\lambda \in$ J .

Another upper bound for the norm of the translated operator (non trivial if $w(A) < \|A\|$ and more accurate for large values of $|\lambda|$) is proved in [7]

$$\|A + \lambda I\| \le w(A) + \sqrt{w^2(A) + |\lambda|^2}.$$

In general this estimate can not be sharpened. According to Proposition 2 from [7] if $A^2 = 0$ then for any c the equality $\|A + \lambda I\| = w(A) + \sqrt{w^2(A) + |\lambda|^2}$ is satisfied. This equality implies $\|A\| = 2w(A)$.

Proposition 7. The equality $w(A) = |\lambda|, \lambda \in \overline{W(A)}$ is satisfied if and only if $w(A) + |z| \le \|A + zI\|$ for all $z, \arg z = \arg\lambda$.

Proof. The condition of this proposition means that $\lim\limits_{|\lambda|\to\infty}\left(\|A+zI\|-|z|\right)=$ $=w(A)$. As the function

$f(x)=\|A+xI\|-x, x\geq 0$ is decreasing, we get $\|A+zI\|\leq w(A)+|z|$.

Proposition 8. The equality $\|A\|=2w(A)$ is satisfied if and only if $\|A+\lambda I\|\leq\|A\|/2+\sqrt{\|A\|^2/4+|\lambda|^2}$ for any $\|A+\lambda I\|\leq\|A\|/2+\sqrt{\|A\|^2/4+|\lambda|^2}$.

Proof. The necessity of the inequality is obvious. Let now the inequality be satisfied. Then taking $\lambda=|\lambda|exp(i\varphi)$, we have

$$f(\phi)=\lim_{|\lambda|\to\infty}\left(|\lambda|-\|A-\lambda I\|\right)=\lim_{|\lambda|\to\infty}\left(|\lambda|-\frac{\|A\|}{2}-\sqrt{\frac{\|A\|^2}{4}+|\lambda|^2}\right)=-\frac{\|A\|}{2}$$

and $w(A)=-\sup\limits_{\varphi\in[0;2\pi)}f(\varphi)$, implying $w(A)\leq\|A\|/2$.

The inverse inequality being valid for any operator, finally we get $w(A)=\|A\|/2$.

By [8], Corollary of Proposition 3 the last condition is satisfied if and only if

$$\bar{W}(A)=D\left(0,\|A\|/2\right).$$

Example. Let

$$D=\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}.$$

As it is well known that for any matrix $\|A\|=\max\left\{eig\left(A^*A\right)\right\}$, where eig is the set of eigenvalues of the matrix. It is easy to check that

$$\|D+\lambda I\|=\max\left\{\sqrt{1+|\lambda|^2+2\operatorname{Re}\lambda},\frac{1}{2}+\sqrt{\frac{1}{4}+|\lambda|^2}\right\}.$$

These expressions are equal on the right branch of the hyperbola, defined by the equation

$9\left(x+\frac{1}{3}\right)^2-3y^2=1$. If λ is a positive number, then $\|D+\lambda I\|=1+\lambda=\|D\|+\lambda$, i.e. the Daugavet equality is satisfied on the real positive semi-axis. Beyond the inner domain, bounded by the right branch of the hyperbola

$$\|D+\lambda I\|=\|D\|/2+\sqrt{\|D\|^2/4+|\lambda|^2}.$$

Note that on the ray $z=re^{i\phi}, 0<r<\infty, \frac{\pi}{3}<|\phi|<\frac{\pi}{2}$ at small values of r one has the first equality, then the second.

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Characterization of Spectraloid and Normaloid Operators

The spectraloid and normaloid operators are characterized in infinite dimensional Hilbert space. It is shown how this description may be modified to settle the finite dimensional case. The ratio between the norm and the numerical radius of a square matrix was a subject of recent investigations. We consider this problem for bounded operators and show that one extremal value is connected with the Daugavet equality and the second is equivalent to an inequality. Possible shapes of the numerical ranges of extremal operators are described.

Л. З. Геворгян

Характеристика спектралоидных и нормалоидных операторов

Характеризируются спектралоидные и нормалоидные операторы, действующие в бесконечномерном гильбертовом пространстве. Показано, как данное описание может быть приспособлено для конечномерного случая. Частное нормы и числового радиуса для квадратной матрицы рассматривалось в недавних исследованиях. Данная проблема изучена для ограниченных операторов; показано, что одно экстремальное значение связано с равенством Даугавета, а другое эквивалентно некоторому неравенству. Описаны возможные формы числовых образов экстремальных операторов.

Լ. Զ. Գևորգյան

Սպեկտրալոիդ և նորմալոիդ օպերատորների բնութագրումը

Բնութագրվում են անվերջ չափանի հիլբերթյան տարածությունում գործող սպեկտրալոիդ և նորմալոիդ օպերատորները: Ցույց է տրվում, թե ինչպես այդ նկարագրությունը կարող է ձևափոխվել, որպեսզի ծառայի նաև վերջավոր չափանի դեպքում: Քառակուսի մատրիցի նորմի և թվային շառավղի քանորդը հետազոտվել է վերջերս կատարված ուսումնասիրություններում: Այստեղ քննարկվում է այդ խնդիրը սահմանափակ օպերատորների համար և ցույց է տրվում, որ մի էքստրեմալ արժեքը կապված է Դաուգավետի հավասարության հետ, իսկ մյուսը համարժեք է որոշակի անհավասարության: Նկարագրված են էքստրեմալ օպերատորների թվային պատկերների հնարավոր տեսքերը:

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