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Some Estimates for Stationary Extended Mean Field Games

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1.Introduction. Mean field games is a recent area of research started by Caines and his coworkers [1, 2], and independently by Pierre Louis Lions and Jean Michel Lasry [3,4,5]. It attempts to understand the limiting behavior of systems involving very large numbers of rational agents which play dynamic games under partial information and symmetry assumptions. Inspired by ideas in statistical physics, these authors introduced a class of models in which the individual player contribution is encoded in a mean field that contains only statistical properties about the ensemble. The literature on mean field games and its applications is growing fast, for a recent survey see [7] and reference therein. Applications of mean field games arise in the study of growth theory in economics [8] or environmental policy [9], for instance, and it is likely that in the future they will play an important role in economics and population models. There is also a growing interest in numerical methods for these problems [9,10]. By some authors there are considered the discrete time, finite state problem, and the continuous time finite state problem [11,12].
Denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$. Let

$$H = H(x,p,m): \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

be a function satisfying appropriate continuity, differentiability and growth conditions. Stationary mean field game is a system of PDE's on $\mathbb{T}^d$. of the form

$$\begin{cases} \Delta u(x) + H(x, Du(x), m) = \overline{H} \\ \Delta m(x) - \operatorname{div}\big(D_p H(x, Du(x), m) m(x)\big) = 0 \end{cases}$$

The previous system of PDE's arise naturally in the study of mean-field games. Not only stationary mean- field games have an independent interest but also, due to results in [12] and [13] concerning the trend to the equilibrium of time dependent mean-field games they encode the asymptotic properties of such problems. A certain type of stationary mean field games can also arise from variational problems: given $H: \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ consider the stochastic Evans-Aronsson problem

$$\inf_{\varphi} \int_{\mathbb{T}^d} e^{\varepsilon\Delta\varphi(x)+H(x,D\varphi(x))} dx$$

The Euler-Lagrange for this functional can be written as

$$\begin{cases} \varepsilon\Delta u(x) + H\big(x, Du(x)\big) = \ln m(x) + \overline{H} \\ \varepsilon\Delta m(x) - \operatorname{div}\big(D_p H\big(x, Du(x)\big) m(x)\big) = 0 \end{cases}$$

In [14] there are considered the so called quasi-variational mean-field games, which are perturbations of mean field games with a variational structure. In this paper we consider an extension of the mean field problem which allows the cost function of a player also to depend on the actions of the rest of the players, that is on the velocity field of players. Denote by $\chi(\mathbb{T}^d)$ the set of continuous vector fields on $\mathbb{T}^d$. Let

$$H: \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$$

We consider the following equation on the d-dimensional torus $\mathbb{T}^d$.

$$\begin{cases} \Delta u(x) + H(x, Du(x), m, V) = \overline{H} \\ \Delta m(x) - \operatorname{div}\big(V(x) m(x)\big) = 0 \\ V(x) = D_p H(x, Du(x), m, V). \end{cases} \tag{1}$$

The unknowns for this problems are $u: \mathbb{T}^d \to \mathbb{R}$ identified with the 1- periodic functions on $\mathbb{R}^d$ whenever it is convenient, a probability measure $m \in \mathcal{P}(\mathbb{T}^d)$, an effective Hamiltonian $\overline{H}$ and the effective velocity field $V \in \chi(\mathbb{T}^d)$. We require m to be a probability measure absolutely continuous with respect to Lebesgue measure. By abuse of notation we use the symbol m both for the measure and the density function. We assume that H is C^2 in the variables x and p and satisfies growth conditions as detailed below:

H is quasi-variational [15], i.e., there exists a concave, increasing function $g: (0, \infty) \to \mathbb{R}$ and

$$H_0: \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$$

continuous in all its variables, such that the following assumption is satisfied:

(A1)

$$|H(x, p, m, V) - H_0(x, p, m, V) + g(m(x))| \le C$$

In the current paper we establish some a-priori estimates for the solutions to this system. This kind of estimates can be later used in proving the existence of the solutions via continuation method.

2. Assumptions. In this section we introduce and discuss the various assumptions that will be needed throughout the paper, in addition to (A1).

(A2) The function $g : (0, \infty) \to \mathbb{R}$ is smooth, strictly increasing and concave, with $tg(t)$ convex in t.
In some results we will consider the following particular choices of g :

a) $g(t) = \ln t$,
b) $g(t) = t^\gamma$, with $0 < \gamma < 1$,

in which case we will refer to them as, respectively, assumption (A2a) or (A2b).

We suppose that for the Hamiltonians $H: \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$ and $H_0: \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$ there exist constants $C, \delta > 0$ such that the following assumptions are satisfied:

(A3) For all $(x, p, m, V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|p|^2 \le C + C\, H_0(x, p, m, V) + \delta \int_{\mathbb{T}^d} |V|^2 dm$$

(A4) The function $H(x, p, m, V) + g(m(x))$ is twice differentiable in variables x, p. For $(x, p, m, V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$ we define the Lagrangian L associated with H as

$$L(x, p, m, V) = -H(x, p, m, V) + pD_pH(x, p, m, V).$$

With this notation we assume further:
(A5) There exists $c > 0$ such that, for all

$$(x, p, m, V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$$

$$L(x, p, m, V) \ge c\, H_0(x, p, m, V) + g(m(x)) - C - \delta \int_{\mathbb{T}^d} |V|^2 dm$$

(A6) For all $(x, p, m, V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|D_pH(x, p, m, V)|^2 \le C + CH_0(x, p, m, V) + \delta \int_{\mathbb{T}^d} |V|^2 dm$$

Another hypothesis concerns the convexity of H in p. We suppose:
(A7) H is uniformly convex in p: there exists $\theta > 0$ such that for all $(x, p, m, V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$, the matrix $D^2_{pp}H(x, p, m, V) - \theta I$ is positive definite, where I is the identity matrix in $\mathbb{R}^d$.

Set

$$\widehat{H}_x = D_x\big[H(x,p,m,V) + \mathrm{g}(\mathrm{m(x)})\big],$$
$$\widehat{H}_{xx} = D_{xx}^2\big[H(x,p,m,V) + \mathrm{g}(\mathrm{m(x)})\big],$$
$$\widehat{H}_{xp} = D_x\left(D_p H(x,p,m,V)\right).$$

(A8) For all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|\widehat{H}_{xx}(x,p,m,V)| \leq C + C\,H_0(x,p,m,V) + \delta\int_{\mathbb{T}^d}|V|^2 dm$$

(A9) For all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|\widehat{H}_{xp}(x,p,m,V)|^2 \leq C + C\,H_0(x,p,m,V) + \delta\int_{\mathbb{T}^d}|V|^2 dm$$

For example, the following Hamiltonian satisfies all those conditions:

$$H(x,p,m,V) = \frac{1}{2}|p|^2 - \alpha p\int_{\mathbb{T}^d} V(y)dm(y) - \mathrm{g}(m(y)),$$

with $\delta = \frac{\alpha^2}{2}$.

3. A-priori estimates. In this section we are going to present various a-priori estimates for the solutions to the quasivariational stationary extended mean-field game equation (1). Hereafter, by a solution to (1) we mean a tuple $(u,m,\mathrm{V},\overline{H})$ where u is m smooth, $m > 0$ such that the equations in (1) are satisfied in the classical sense. Recall that by abuse of notation we denote both the measure m and its density function by the same letter, so by saying m is smooth and positive, we mean that the density of measure m is smooth and positive.

In this section we use ideas and techniques from [15] with some modifications to allow for the dependence of H on V.

Proposition 1. *Assume conditions (A1)-(A5) hold and* $(u,m,\mathrm{V},\overline{H})$ *is a solution to the system (1). Then there exists* $C > 0$*, such that*

$$|\overline{H}| \leq C + C\,\delta\int_{\mathbb{T}^d}|V|^2 dm.$$

Corollary 2. *Assume conditions (A1)-(A5) hold and* $(u,m,\mathrm{V},\overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$ *such that*

$$\int_{\mathbb{T}^d} H_0(x,Du(x),m,V)dx \leq C + C\,\delta\int_{\mathbb{T}^d}|V|^2 dm.$$

Corollary 3. *Assume conditions (A1)-(A5) hold and* $(u,m,\mathrm{V},\overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$ *such that*

$$\int_{\mathbb{T}^d} H_0(x,Du(x),m,V)dm \leq C + C\,\delta\int_{\mathbb{T}^d}|V|^2 dm.$$

Proposition 4. *Assume conditions (A1)-(A6) hold and* $(u, m, \mathrm{V}, \overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$ *such that*

$$\|\sqrt{m}\|_{H^1(\mathbb{T}^d)} \leq C + C\,\delta\int_{\mathbb{T}^d}|V|^2 dm.$$

Proposition 5. *Assume conditions (A1)-(A6) hold and* $(u, m, \mathrm{V}, \overline{H})$ *is a solution to the system (1). Then, there exists* $\delta_0 > 0$ *such that for any* $\delta \in [0, \delta_0]$ *we have*

$$\int_{\mathbb{T}^d}|V|^2 dm \leq C.$$

Combining Proposition 5 with Propositions 1, 4 and Corollaries 2, 3 we get:

Corollary 6. *Assume conditions (A1)-(A6) hold and* $(u, m, \mathrm{V}, \overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$ *such that for any* $\delta \in [0, \delta_0]$ *we have*

$$|\overline{H}| \leq C,$$
$$\int_{\mathbb{T}^d} H_0(x, Du(x), m, V)dx \leq C,$$
$$\int_{\mathbb{T}^d} H_0(x, Du(x), m, V)dm \leq C,$$
$$\|\sqrt{m}\|_{H^1(\mathbb{T}^d)} \leq C.$$

Proposition 7. *Assume conditions (A1)-(A9) hold and* $(u, m, \mathrm{V}, \overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$, *such that for any* $\delta \in [0, \delta_0]$ *we have*

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx \leq C,\ \int_{\mathbb{T}^d}|D^2 u|^2 dm \leq C.$$

Corollary 8. *Assume conditions (A1), (A2a), (A3)-(A9) hold, and* $(u, m, V, \overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$ *such that for any* $\delta \in [0, \delta_0]$ *we have*

$$\int_{\mathbb{T}^d} m^{\frac{2^*}{2}} dx \leq C,$$

and

$$\int_{\mathbb{T}^d}|Du|^4 dx \leq C,$$

where $2^* = \frac{2d}{d-2}$ is the Sobolev conjugate of 2.

Corollary 9. *Assume conditions (A1), (A2b), (A3)-(A9) hold, and* $(u, m, \mathrm{V}, \overline{H})$ *is a solution to the system (1). Then, there exists* $C > 0$ *such that for any* $\delta \in [0, \delta_0]$ *we have*

$$\int_{\mathbb{T}^d} m^{\frac{2^*}{2}(\gamma+1)} dx \leq C\ .$$

Furthermore, If $2\gamma + 1 \leq \frac{2^*}{2}(\gamma + 1)$, *then*

$$\int_{\mathbb{T}^d} |Du|^4 dm \leq C.$$

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Some Estimates for Stationary Extended Mean Field Games

In the paper we consider quasivariational mean field games system with additional dependence on a velocity field of the players. We obtain certain a-priori estimates for the solutions to that system.

В. К. Восканян

Некоторые оценки для обобщенных игр среднего поля

Рассматривается система уравнений игр среднего поля квазивариационного типа с дополнительной зависимостью от поля скоростей игроков. Получены некоторые априори оценки для решений этой системы.

Վ. Կ. Ոսկանյան

Որոշ գնահատականներ ընդհանրացված ստացիոնար միջին դաշտի խաղերի համար

Դիտարկվում է քվազիվարիացիոն տիպի միջին դաշտի խաղերի հավասարումների համակարգ՝ խաղացողների արագությունների դաշտից հետևյալ կախվածությամբ: Ստացվել են ապրիորի գնահատականներ այդպիսի համակարգի լուծումների համար:

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