A. Yu. Shahverdian

Minimal Lie Algebra, Fine limits, and Dynamical Systems

(Submitted by academician V. S. Zakaryan 14/XI 2011)

Key words: discrete dynamical systems, Lie algebras, fine sets, fine attractors

1. Introduction.

This paper advances the suggested in [1]–[4] difference analysis which is a method for analyzing discrete dynamical systems. This method is based on studying the higher absolute differences taken from progressive terms of system's orbit. This is motivated by an observation [5] that some natural systems (e.g., the visual cortex) process the information contained in signal's higher difference structure. A minimal Lie algebra considered in this paper provides us with the axiomatic basis for difference analysis. The identity $x + x \equiv 0$, which is a basic one in the Lie algebras, leads us to some exotic arithmetic, maps, and dynamics. The situation considered is opposite (in axiomatic sense) to so-called idempotent or tropical analysis [6] where another exotic identity $x + x \equiv x$ is postulated.

In next Section 2 the difference analysis is briefly described and a minimal Lie algebra is defined. In Section 3 we are interested in the topological aspect of the suggested algebra – to determine the limiting difference orbits (as the order of differences tends to ∞) we involve a version of the thinness from the probabilistic potential theory. In last Section 4 a difference-shift transform and Bernoulli maps are studied.

2. Difference analysis and minimal Lie algebra.

2.1. ***Difference analysis.*** In difference analysis a given orbit is decomposed into the sign and magnitude components which are studied independently. The sign component $S = (S_m)_{m=1}^{\infty}$

reflects the alternation in monotony (increase/decrease) of higher absolute differences taken from the successive terms of the orbit. The magnitude (or, height) component $H = (H^{(m)})_{m=1}^\infty$, which consists of these absolute differences, does not depend on S_m.

Let us present some formal definitions. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$ $(0 \leq x_n \leq 1)$ be an infinite numerical sequence – this can be some time series, results of an experiment, or an orbit of some map. We consider m-th order absolute differences:

$$H_n^{(0)} = x_n, \quad H_n^{(m+1)} = |H_{n+1}^{(m)} - H_n^{(m)}| \qquad (m \geq 0, \ n \geq 1) \tag{1}$$

and define m-th difference sequence as $H^{(m)} = (H_n^{(m)})_{n=1}^\infty$. We also let

$$S_m = (\delta_1^{(m)}, \delta_2^{(m)}, \ldots, \delta_n^{(m)}, \ldots) \quad \text{where} \quad \delta_n^{(m)} = \begin{cases} 1, & H_{n+1}^{(m)} \geq H_n^{(m)} \\ 0, & H_{n+1}^{(m)} < H_n^{(m)}. \end{cases} \tag{2}$$

The S_m and $H^{(m)}$ are called m-th S- and H-components of $\mathbf{x}$. Instead of binary sequence in Eq. (2) one can also deal with the corresponding real numbers $S_m = 0.\delta_1^{(m)}\delta_2^{(m)} \ldots$ (binary expansion; the same notation as in (2)). Thus, given time series (orbit) $\mathbf{x}$ we define and study two others, the sign and magnitude series $(S_m)_{m=1}^\infty$ and $(H^{(m)})_{m=1}^\infty$.

The S-component extracts from $\mathbf{x} = (x_n)_{n=1}^\infty$ some binary constituents S_m. One can prove that for arbitrary $\mathbf{x}$ the H-component also contains arbitrarily long binary samples. Hence, for many cases the difference orbits are reduced to a "minimal" form of binary sequences – namely with the aim of studying this "minimal" case a simplified version of the Lie algebra in next Section is introduced.

2.2. *Minimal Lie algebra.* The binary operation of taking the absolute difference $\xi \oplus \eta = |\xi - \eta|$ between two binary variables $\xi, \eta \in \{0, 1\}$ satisfies the identity $\xi \oplus \xi = 0$ postulated in Lie algebras as well as the group-theoretical axioms (Eq. (3)); generalizing this observation we consider some minimal abstract Lie algebras.

Let X be an abstract set on which a binary operation denoted $[x, y]$ be defined. It is assumed that this operation satisfies relations:

$$[x, [y, z]] = [[x, y], z], \quad [x, y] = [y, x], \quad [x, 0] = x, \quad [x, x] = 0 \tag{3}$$

that is, $G = (X, [,])$ is an abstract commutative group on X. For $x \in X$ and binary $\alpha \in \{0, 1\}$ we define a multiplication: $\alpha x = 0$ if $\alpha = 0$ and $\alpha x = x$ if $\alpha = 1$. Since our main interest is the last relation in (3) which is a basic one in Lie algebras, we call the group G with assigned on it binary multiplication αx a minimal Lie algebra.

We extend the binary bracket (3) to n-ary version $[x_1, \ldots, x_n]$: if for some $n \geq 3$ the $(n-1)$-ary bracket is already defined, then we let

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], [x_2, \ldots, x_n]].$$

As a consequence of the relation $[x, x] = 0$ in (3) it follows (next Proposition 1) that n-ary bracket can be expressed by a binary version $\mathbb{P}$ of the Pascal triangle of binomial coefficients, $\mathbb{P} = (\alpha_{i,k})_{i,k}$ where $0 \leq i \leq k$, $k \geq 1$, and $\alpha_{i,k} \in \{0, 1\}$. The first line $(k = 1)$ of $\mathbb{P}$ consists of

a single number 1 denoted as $\alpha_{0,1}$ or $\alpha_{1,1}$, and its every k-th line $(k \geq 2)$ $\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{k,k}$ is determined recurrently: we suppose $\alpha_{0,k} = \alpha_{k,k} = 1$ and then let

$$\alpha_{i,k} = \alpha_{i-1,k-1} \oplus \alpha_{i,k-1} \quad (1 \leq i \leq k-1), \quad \text{that is} \quad \alpha_{i,k} = \begin{cases} 0, & \binom{k}{i} \text{ is even} \\ 1, & \binom{k}{i} \text{ is odd.} \end{cases}$$

Proposition 1. *Let $x_0, \ldots, x_n \in X$ and $z_m = [x_0, \ldots, x_m]$ $(0 \leq m \leq n)$. Then it follows:*

$$z_n = [\alpha_{0,n}x_0, \alpha_{1,n}x_1, \cdots, \alpha_{n,n}x_n], \quad x_n = [\alpha_{0,n}z_0, \alpha_{1,n}z_1, \cdots, \alpha_{n,n}z_n].$$

We assume that there is a functional μ,

$$\mu : X \setminus \{0\} \to \mathbb{R}^+ \quad \text{such that} \quad \mu([x_0, \ldots, x_n]) = \sum_{i=0}^{n} \alpha_{i,n}\mu(x_i) \tag{4}$$

(here, $\mu(0)$ is not defined since (4) with equal x_i can lead to some contradictions) and assign a topology on X by defining: if $x \neq 0$ then $x_n \to x$ (x_n converges to x as $n \to \infty$) if $\mu(x_n) \to \mu(x)$, and $x_n \to 0$ if $\mu(x_n) \to 0$. Then it follows that $G = G(X, [,], \mu)$ is a topological group. One can consider the direct products of such algebras (e.g., for the aims of the multidimensional difference analysis). Let $G_i = G(X_i, [,]_i, \mu_i)$, $1 \leq i \leq n$, $n \geq 2$ be some minimal Lie algebras; then one can construct such algebra $\mathbf{G} = G_1 \times \cdots \times G_n$ on cartesian product $\mathbf{X} = X_1 \times \cdots \times X_n$ by assuming that $[\mathbf{x}, \mathbf{y}] = ([x_1, y_1]_1, \ldots, [x_n, y_n]_n)$ for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and that $\boldsymbol{\mu} : \mathbf{X} \to \mathbb{R}^n$ is given by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$.

2.3. ***Independent random binary processes.*** We present an example of minimal algebra defined on the collection X of pairwise independent binary random variables $\xi \in \{0, 1\}$. For $\xi, \eta \in X$ we consider a variable $\xi \oplus \eta$ $(= [\xi, \eta])$, whose distribution of probabilities coincides with the distribution of the absolute difference $|\xi - \eta|$. We assign the probabilities as $P(\xi = \lambda) = \frac{1}{2}(1 \pm (-1)^\lambda \pi)$ where $\lambda \in \{0, 1\}$ and $\pi \in (0, 1)$ and denote $\pi = \pi(\xi)$ (to avoid some formal complications we assumed $\pi \neq 0, 1$). The $\mu(\xi) = -\ln \pi(\xi)$ is a functional of the type (4) (see next Theorem 1), and hence, we obtain a probabilistic example $G = G(X, \oplus, \mu)$ of the minimal algebra.

In the following we consider the absolute differences taken from a given random process $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ where ξ_n are independent and take the values 0 and 1 with positive probabilities – such $\boldsymbol{\xi}$ (but not their difference series) are studied, e.g., in a paper by Marsaglia [7]. The variables $\xi_n \oplus \xi_{n+1}$ also take binary values with some positive probabilities, and the difference (of order 1) process $\boldsymbol{\xi}^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)}, \ldots, \xi_n^{(1)}, \ldots)$ is defined as independent process where the distribution of $\xi_n^{(1)}$ coincides with the distribution of $\xi_n \oplus \xi_{n+1}$. The k-th order $(k \geq 1)$ difference process $\boldsymbol{\xi}^{(k)}$ is defined as 1st order difference taken from $\boldsymbol{\xi}^{(k-1)}$. Our assumption that for every $0 \leq k \leq +\infty$ the process $\boldsymbol{\xi}^{(k)}$ is independent yields that $\pi(\xi_n^{(k)})$ are determined by kth line of $\mathbb{P}$:

Theorem 1. *For $n, k \geq 1$ the following relations are true:*

$$\xi_n^{(k)} = [\xi_n, \ldots, \xi_{n+k}], \quad \ln \pi([\xi_n, \ldots, \xi_{n+k}]) = \sum_{0 \leq i \leq k} \alpha_{i,k} \ln \pi(\xi_{n+i}). \tag{5}$$

3. **Fine limits in minimal Lie algebras**.

We are interested in the topological aspect of the algebras $G = (X, [,], \mu)$ – we consider the limits of infinite sequences from X. In potential theory the Wiener criterion and fine sets are used for studying the limits of potentials (e.g., the Cartan theorem on quasi-continuity of the Newton potential [8, 9]). Following this, we assign some "fine" sets ($\mathfrak{F}$-sets) in natural series $\mathbb{N}$ and consider fine convergence ($\mathfrak{F}$-convergence) and fine limits ($\mathfrak{F}$-lim) of difference orbits. We use potential theory terminology (fine or thin sets, etc) because of formal similarity of next relation (7) with the Wiener criterion in the probabilistic potential theory [10, 11].

We consider the following binary codes of numbers $k \in \mathbb{N}$ – it is a vector $(s_0, s_1, \ldots, s_{p-1})$ with binary components $s_i \in \{0, 1\}$ (bits) for which $k = \sum_{i=0}^{p} s_i 2^i$ and $s_p = 1$. Let $w(k) = \sum_{i=0}^{p-1} s_i$ be the number of units (the weight) in the code of k. For $e \subset \mathbb{N}$ we define

$$\mathcal{C}(e) = \sum_{k \in e} w(k) \tag{6}$$

and call $\mathcal{C}(e)$ the capacity of e. For $E \subseteq \mathbb{N}$ we denote $E_n = E \cap \{k \in \mathbb{N} : 2^n \leq k < 2^{n+1}\}$.

Definition 1. *A set $E \subset \mathbb{N}$ is called a fine set ($\mathfrak{F}$-set) if the relation*

$$\sum_{n=1}^{\infty} 2^{-n} \mathcal{C}(E_n) < \infty \tag{7}$$

holds. An infinite sequence $x_n \in X$ is called $\mathfrak{F}$-convergent to x,

$$\mathfrak{F}-\lim_{n \to \infty} x_n = x, \text{ if there is an } \mathfrak{F}-\text{set } E \text{ such that } \lim_{\substack{n \to \infty \\ n \in \mathbb{N} \setminus E}} \mu(x_n) = \mu(x). \tag{8}$$

The $\mathcal{C}$ is an additive measure on subsets of $[2^n, 2^{n+1})$ and $\mathcal{C}([2^n, 2^{n+1})) = 2^n$. We consider the sets $B_n(s) = \{2^n \leq k < 2^{n+1} : w(k) \leq s\}$ where $0 \leq s \leq n$. Since for some cases $\mathcal{C}(B_n(s)) \leq const.\mathcal{C}(\partial B_n(s))$ (when n is large and s is small; for such cases the $\mathcal{C}(B_n(s))$ can be substituted by $\mathcal{C}(\partial B_n(s))$ which is equal to $\binom{n}{s}$) what is a characteristic property of classical capacities (with $const. = 1$, e.g., [10]), we called $\mathcal{C}$ a capacity.

It follows from (6) that the capacity $\mathcal{C}$ concerns several notions in information theory. To demonstrate this we identify a number $k \in [2^n, 2^{n+1})$ (the segment of natural numbers, $2^n \leq k < 2^{n+1}$) with a vertex $(s_0, s_1, \ldots, s_{n-1})$ (the binary code of k) of n-dimensional cube $[0, 1]^n$. If $(s_0, s_1, \ldots, s_{n-1})$ and $(s'_0, s'_1, \ldots, s'_{n-1})$ are the codes of k and k', then the Hamming distance between k and k' is the number of such i for which $s_i \neq s'_i$. By Eq. (6) the capacity $\mathcal{C}$ of $B_n(s)$ is equal to the following sum of binomial coefficients: $\sigma(n, s) = \sum_{i=0}^{s} \binom{n}{i}$. The cardinality of $B_n(s)$ (the Hamming volume) is also equal to $\sigma(n, s)$; the σ coincides also with some other quantities in coding theory mentioned in next Proposition (see, e.g., [13] for details and definitions; for estimates of σ by Shannon function see [14]). In next formulation, $V_H(e)$ is the Hamming volume of e, b_H is the Hamming bound from coding theory (the $b_H(n, s)$ is an upper bound for the size of binary s-error corrected codes of length n), and $H(x) = x \log_2 x + (1-x) \log_2(1-x)$ is the Shannon entropy function (it is assumed for its argument in (9) that $s/n < 1/2$).

Proposition 2. *The $B_n(s)$ is a ball of radius s in the Hamming metric (the Hamming ball) centered at 2^n and the following relations*

$$\mathcal{C}(B_n(s)) = V_H(B_n(s)) = 2^n(b_H(n,s))^{-1} = 2^{nH(s/n)+o(n)} \tag{9}$$

hold. Particularly, it follows that the union $E = \bigcup_{n=1}^{\infty} B_n(s_n)$ is an $\mathfrak{F}$–set if and only if

$$\sum_{n=1}^{\infty} (b_H(n,s_n))^{-1} < \infty.$$

A computation shows that there exists a union of balls $E = \bigcup_{n=1}^{\infty} B_n(s_n)$ whose radii s_n grow to ∞ and such that E is an $\mathfrak{F}$-set. Hence, every such E with upper bounded radii s_n is also an $\mathfrak{F}$-set. This explains the reason of our Definition 1 – it is allowed by this Definition that some small sets E of indices $k \in \mathbb{N}$ can be neglected when determining the $\mathfrak{F}$-limits (cp. Eq. (8)). For example, if E is a union of balls $B_n(s_n)$ whose radii $s_n \leq s$, then for every $k \in E$ the $\pi(\xi_n^{(k)})$ (Section 2.3) is a product of lesser than s different $\pi(\xi_i)$, what implies that as a rule, their limit as k growth to ∞ remaining in E, does not exist. E.g., one can refer to identically distributed $\boldsymbol{\xi}$ – here, $\pi(\xi_i) \equiv \pi$, $0 < \pi < 1$ and $\pi(\xi_n^{(k)}) \equiv \pi^{2^{w(k)+1}}$ (a self-similar structure of the triangle $\mathbb{P}$ yields that the number of units in k-th line of $\mathbb{P}$ is equal to $2^{w(k)+1}$).

The discussed is similar to metrical statements on covering the classical fine sets by (euclidean and non-euclidean) balls (e.g., [9]). We formulate another related metrical result from [2, 4]. For $x \in [0,1]$ we consider infinite binary expansions $x = 0.x_1, x_2, \ldots$ and let for $k \geq 2$ the E_k be the collection of such $x \in [0,1]$ for which every segment $x_i, \ldots, x_{i+k}$ of the expansion contains both symbols 0 and 1.

Theorem 2. *For every $k \geq 2$ the Hausdorff dimension $D(E_k)$ of the set E_k satisfies the following relation:*

$$\sum_{n=1}^{k} 2^{-nD(E_k)} = 1.$$

For q-adic version of this Theorem, see [2, 4].

Definition 2. *Let $G = G(X, [,], \mu)$ be given, $\mathbf{x} = (x_1, x_2, \ldots)$, $x_n \in X$ be infinite sequence, $x_n^{(k)} = [x_n, \ldots, x_{n+k}]$, and x_n^{∞} be $\mathfrak{F}$-limit of the sequence $x_n^{(k)}$ as $k \to \infty$. The $\boldsymbol{x}^{\infty} = (x_1^{\infty}, x_2^{\infty}, \ldots)$ is called the final difference sequence for $\boldsymbol{x}$.*

Thus, it follows from (4) that if for $\mathbf{x} = (x_1, x_2, \ldots)$ the $\boldsymbol{x}^{\infty}$ exists, then for every n

$$\mathfrak{F}-\lim_{k\to\infty} \sum_{i=0}^{k} \alpha_{i,k}\mu(x_{n+i}) = \mu(x_n^{\infty}).$$

E.g., it follows from (5) that for a given random process $\boldsymbol{\xi}$ the differences $\boldsymbol{\xi}^{(k)}$ converge to $\boldsymbol{\xi}^{\infty}$ iff for every n the $\pi(\xi_n^{(k)})$ is $\mathfrak{F}$-convergent to some numbers $s_n \in [0,1]$; then $\pi(\xi_n^{\infty}) = s_n$.

Theorem 3. *If $\boldsymbol{\xi}$ is identically distributed binary process (the Bernoulli trials), then $\boldsymbol{\xi}^{\infty}$ is the symmetric equi-distributed process.*

The next Theorem relates to arbitrary $G = (X, [,], \mu)$, usual limits, and ω-sets (cluster sets); we remind that a set is called a perfect set if it is closed and its every point is also its limiting point. In applications this Theorem can concern the conservative systems as well as can explain the emergence of the Cantorian structure in the attractors of dissipative systems (see Theorem 7 for an example):

Theorem 4. *Let $x_n \in X$ be infinite sequence and $z_n = [x_0, \ldots, x_n]$. If $\mu(x_n) = const. > 0$, then ω-set of the sequence $\mu(z_n)$ is a discrete countable closed set. If $\mu(x_n)$ is monotone and decreases to 0, then ω-set of the $\mu(z_n)$ is a nontrivial perfect set.*

4. Application to dynamical systems.

We define a difference-shift transform ([2, 4]) and present some results on so-called Bernoulli maps. We first consider arbitrary groups $G = (X, [,])$ defined by (3) and formulate the following (the Proposition 4 concerns arbitrary minimal algebras $G = (X, [,], \mu)$:

Definition 3. *Let $T : X \to X$ be a map. The map $\hat{T} : X \to X$ defined as $\hat{T}x = [x, Tx]$ is called algebraically conjugate to T. A map T is called commutative if $T\hat{T} = \hat{T}T$.*

Proposition 3. *Let $T : X \to X$ be commutative. Then for $x \in X$ and $n \in \mathbb{N}$*

$$[x, Tx, \ldots, T^n x] = \hat{T}^n x, \quad [x, \hat{T}x, \ldots, \hat{T}^n x] = T^n x. \tag{10}$$

Proposition 4. *Let $T : X \to X$ be commutative and $x_n = T^n x$ be an orbit. Then if x_1^∞ exists, then for every $n \geq 1$ the x_n^∞ also exists, and the final orbit is: $x_n^\infty = T^n x_1^\infty$.*

The algebraic conjugation differs from the topological one: e.g., the Bernoulli shift B (defined in next Section) is topologically conjugate to so-called tent and 4-logistic maps [12], but B is not algebraically conjugate to them.

4.1. ***A difference-shift map.*** We consider the Bernoulli shift $B : x \mapsto \{2x\}$ (dyadic or bit-shift map [12, 16]; $\{.\}$ and $[.]$ denote the fractional and entire parts of a positive number, $a = \{a\} + [a]$) and the difference-shift map M ([2, 4]), both are defined on the segment $[0, 1]$ (their graphs on Fig. 1 are presented). In binary notation these maps are defined as follows:

$$B : 0.\omega_1 \omega_2 \ldots \mapsto 0.\omega_2 \omega_3 \ldots \quad \text{and} \quad M : 0.\omega_1 \omega_2 \ldots \mapsto 0.(\omega_1 \oplus \omega_2)(\omega_2 \oplus \omega_3) \ldots. \tag{11}$$

The shift M is the sum of the identical map $Ex = x$ and the shift B with respect to some exotic arithmetic, $M = E \dagger B$. This arithmetic is the following: considering binary expansion,

$$\text{for } a = 0.\varepsilon_1 \varepsilon_2 \ldots \text{ and } b = 0.\delta_1 \delta_2 \ldots \text{ we define } a \dagger b = 0.(\varepsilon_1 \oplus \delta_1)(\varepsilon_2 \oplus \delta_2) \ldots. \tag{12}$$

Considering binary expansion of natural numbers: $c(m) = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_q)$ where $m = \sum_{i=0}^q \varepsilon_i 2^i$ and $\varepsilon_i \in \{0, 1\}$, for $a, b \in \mathbb{N}$ we define: if $c(a) = (\varepsilon_0, \ldots, \varepsilon_n)$, $c(b) = (\delta_0, \ldots, \delta_k)$, then $c(a \dagger b) = (\varepsilon_0 \oplus \delta_0, \ldots, \varepsilon_k \oplus \delta_k, \varepsilon_{k+1}, \ldots, \varepsilon_n)$ (we assumed $n \geq k$). For $a, b \in \mathbb{R}^+$ the $a \dagger b \in \mathbb{R}^+$ is a real number for which $[a \dagger b] = [a] \dagger [b]$ and $\{a \dagger b\} = \{a\} \dagger \{b\}$; for example, $5 \dagger 1 = 4$, $5 \dagger 3 = 6$, $6 \dagger 3 = 5$. It is clear how this arithmetic is extended to arbitrary real numbers from $\mathbb{R}$. This determines an additive commutative group $(\mathbb{R}, \dagger)$ with the arithmetic (12) which is not isomorphic to the
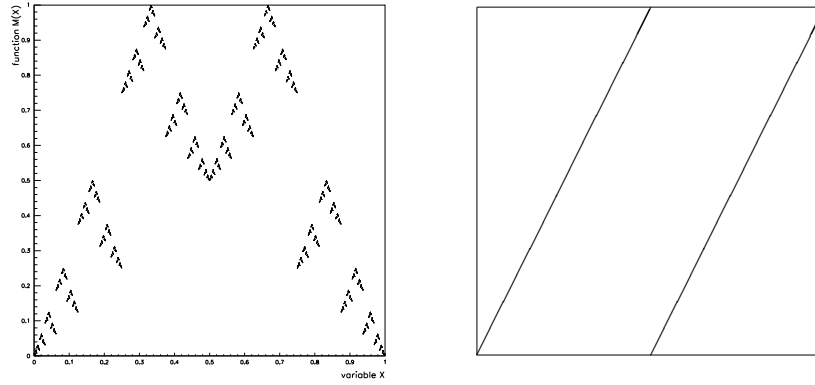
FIGURE 1. The "fractal" letter "M" ([2, 4]) – the graph of the difference shift M, and the graph of the Bernoulli shift B. These maps are algebraically conjugate to each other: $[x, Mx] = Bx$ and $[x, Bx] = Mx$.

standard additive group $(\mathbb{R}, +)$ of real numbers. In next Proposition the group $G = ([0, 1], \dagger)$ with the arithmetic (12) is considered:

Proposition 5. *For the maps B and M the relations $\hat{B} = M$, $\hat{M} = B$, $BM = MB$ hold, and hence, they are algebraically conjugate each other and both are commutative: $B\hat{B} = \hat{B}B$, $M\hat{M} = \hat{M}M$. In addition, Eq. (10) gives the following identities:*

$$[x, Bx, \ldots, B^n x] = M^n x, \quad [x, Mx, \ldots, M^n x] = B^n x. \tag{13}$$

The Eq. (13) and Birkhoff ergodic theorem (one can prove that the Lebesgue measure on $[0, 1]$ is invariant measure for B and for M) imply the "a.e."-convergence ("almost every") of the mean values of difference series $H^{(m)}$ (we denote them $H_B^{(m)}$ and $H_M^{(m)}$) for these maps:

Proposition 6. *There is a positive constant C such that for a.e. $x \in [0, 1]$ the relation*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} H_B^{(m)}(x) = C \qquad \left(= \int_0^1 B(x)dx = 1/2\right)$$

holds. The same is true for the map M.

Let us consider the topological entropy of the map M. The topological entropy $h(T)$ of a map $T : [0, 1] \to [0, 1]$ concerns the ability of T to transform a given segment $\Delta \subseteq [0, 1]$ into a number of smaller ones (for strict definition see, e.g., [15, 17]). Theorem 6 follows from Theorem 5 and the Bowen lemma on periodic orbits ([17], Ch. 5.4).

Theorem 5. *Let $p \geq 3$ and a number $0 < x < 1$ be binary-rational,*

$$x = 0.s_0 s_1 \ldots s_{p-1} s_p 00 \ldots \qquad (s_0, s_1, \ldots s_{p-1} \in \{0, 1\}, \ s_p = 1, s_{p+i} = 0 \text{ for } i \geq 1).$$

Then the orbit $(M^m x)_{m \geq 0}$ is periodic with the period $T = 2^{[\log_2(p-1)]}$.

Theorem 6. *The topological entropy of the map M is positive, $h(M) > 0$.*

The difference orbits of M coincide with its iterates, $S_m = H^{(m)} = M^m$. As noticed in Section 2.1, such a situation is quite common, and hence, the shift M is quite universal when one applies difference analysis. In literature, a universality of the Bernoulli shift B in chaotic dynamics is also noted [12].

4.2. **_Bernoulli maps._** Let us denote $\Omega = [0, 1]$, $X = 2^\Omega$ and for $A, B \in X$ define a bracket $[A, B] = A \triangle B$ where $\triangle$ is the symmetric difference: $A \triangle B = (A \setminus B) \bigcup (B \setminus A)$, Then Eq. (3) is satisfied and hence the group $G = (X, \triangle)$ is determined. We define the multiplication: $0A = \emptyset$ and $1A = A$ and present the following formula:

$$[A_0, \ A_1, \ \ldots, \ A_n] = \alpha_{0,n} A_0 \triangle \alpha_{1,n} A_1 \triangle \cdots \triangle \alpha_{n,n} A_n.$$

Let a positive Borelian measure $\mathfrak{m}$ on Ω be given, $\mathfrak{m}(\Omega) = 1$. We consider Bernoulli maps: a map $T : \Omega \to \Omega$ is called the Bernoulli map (with respect to $\mathfrak{m}$) if for every measurable $A \subseteq \Omega$ the relation $\mathfrak{m}(A \cap TA) = \mathfrak{m}(A)\mathfrak{m}(TA)$ holds. For $A \in X$ one can define a random variable $\xi_A = \xi_A(x)$ which takes the value 1 if $x \in A$ and 0 if $x \in A^C$ and whose distribution of probabilities is: $P(\xi_A = 1) = \mathfrak{m}(A)$ and $P(\xi_A = 0) = 1 - \mathfrak{m}(A)$. It follows that a map $T : \Omega \to \Omega$ is a Bernoulli map iff for every $A \subset \Omega$ of positive $\mathfrak{m}$-measure the variables ξ_A and ξ_{TA} are independent. Then $\mu(A) = -\ln|2\mathfrak{m}(A) - 1|$ is of the type (4) (cp. Section 2.3). Let for a map T and $A \subseteq \Omega$ the A_T^∞ denotes the following fine difference attractor

$$A_T^\infty = \mathfrak{F} - \lim_{n \to \infty} [A, \ TA, \ \ldots, \ T^n A]$$

and $E_A(T)$ denotes the ω-set of the sequence $\mathfrak{m}(\hat{T}^n A)$. The next Theorem provides us with a topological criterion for a measure $\mathfrak{m}$ on Ω to be an invariant measure for a given Bernoulli map. It also asserts that it is either $A_T^\infty = \Omega$ or (under an assumption on monotony) $A_T^\infty = \emptyset$ (in both cases – up to a set of zero $\mathfrak{m}$-measure):

Theorem 7. *Let T be a Bernoulli map on $(\Omega, \mathfrak{m})$ and $A \subseteq \Omega$ be a set of positive $\mathfrak{m}$-measure. If $\mathfrak{m}(T^n A) = \mathfrak{m}(A)$ for every n, then $E_A(T)$ is a discrete countable closed set and $\mathfrak{m}(A_T^\infty) = 1$. If $\mathfrak{m}(T^n A)$ is monotone and decreases to 0 as n growth to ∞, then $E_A(T)$ is a non-trivial perfect set and $\mathfrak{m}(A_T^\infty) = 0$.*

Institute for Informatics and Automation Problems of NAS RA

State Engineering University of Armenia

Email: svrdn@yerphi.am

A. Yu. Shahverdian

Minimal Lie Algebra, Fine Limits, and Dynamical Systems

The paper concerns a method for analyzing discrete dynamical systems, which emphasizes the orbits' higher difference structure. An abstract minimal Lie algebra, which provides us with the axiomatic basis for such analysis, is introduced. The fine sets and limits, defined by means of Wiener criterion (in probabilistic potential theory) type relation, are considered. Some connections with coding theory are discussed. A difference-shift map is defined and its relation to Bernoulli shift is considered. A topological criterion for a measure to be an invariant measure for a given Bernoulli map is established and a result on fine attractors is presented.

А. Ю. Шахвердян

Минимальная алгебра Ли, тонкие пределы, и динамические системы

Ñòàòüÿ êàñàåòñÿ ì åòî äà àí àëèçà äèñêðåòí ûõ äèí àì è÷åñêèõ ñèñòåì , î ñí î âàí í î ro í à ðàññì î òðåí èè âûñøåé ðàçí î ñòí î é ñòðóêòóðû î ðáèò ñèñòåì û. Âàî äèòñÿ àáñòðàêòí àÿ ì èí èì àëüí àÿ àëãåáðà Ëè, êî òî ðàÿ ÿàëÿåòñÿ àêñèî ì àòè÷åñêì áàçèñî ì äëÿ Òàêî ãî àí àëèçà. Ðàññì î òðåí û òî í êèå ì í î æåñòâà è ï ðåäåëû, ï ï ðåäåëÿåì ûå ï ï ñðåäñòâî ì óñëî âèé òèïà êðèòåðèÿ Âèí åðà â âåðî ÿòí î ñòí î é Òåî ðèè ï ï òåí öèàëà. Î áñóæäàþòñÿ ñàÿçè ñ Òåî ðèåé êî äèðî âàí èÿ. Âåâåäåí î î òî áðàæåí èå ðàçí î ñòí î ãî ñäâèãà è ðàññì î òðåí û ñâÿçè ñî ñäâèãî ì Áåðí óëëè. Ï ðèâåäåí Òî ï í ëèàå÷åñêié êðèòåðèé äëÿ èí âàðèàí Òí î ñòè ì åðû äëÿ çàäàí í î ãî î òî áðàæåí èÿ Áåðí óëëè è ñôî ðì óëèðâàí ðåçóüüòàò î òî í êèõ àòòðàêòî ðàõ Òàêèõ î òî áðàæåí èé.

Ա. Յու. Շահվերդյան

Լիի մինիմալ հանրահաշիվ, նուրբ սահմաններ և դինամիկ համակարգեր

Հոդվածը վերաբերվում է տարբերությունների անալիզին, որը դիսկրետ դինամիկ համակարգերի հետազոտության նոր մեթոդ է: Դիտարկվում է Լիի մինիմալ հանրահաշիվ, որը աքսիոմատիկ հիմք է տարբերությունների անալիզի համար: Դիտարկվում են նուրբ բազմություններ, որոնք սահմանվում են հավանակական պոտենցիալի տեսության Վիների տիպի պայմանի միջոցով: Քննարկվում է կապը կոդավորման տեսության հետ: Սահմանվում է տարբերական արտապատկերում և ուսումնասիրվում է նրա կապը Բերնուլիի դիադիկ արտապատկերման հետ: Տրվում է չափի ինվարիանտության տոպոլոգիական պայման և բերվում է նուրբ ատտրակտորներին վերաբերվող թեորեմ:

References

[1] *Shahverdian A. Yu., Apkarian A. V.* – Fractals. 1999. V.7, P. 93-103.

[2] *Shahverdian A. Yu.* – Fractals. 2000. V.8, N.1, P. 49-65.

[3] *Shahverdian A. Yu., Apkarian A. V.* In: Proc. V-th Intern. Congress Math Modeling, JINR, Dubna, Russia, 2002.

[4] *Shahverdian A. Yu., Apkarian A. V.* – Comm. Nonlin. Sci. & Comp. Simul. 2007. V.12, N.1, P. 233–242.

[5] *Miller K. D. In: Domany E., van Hemmen J. L., Schulten K. (Eds).* Models of Neural Networks. V. III, Springer, N.Y., 1996.

[6] *Litvinov G. L., Maslov V. P., Sergeev S.N. (Eds).* Idempotent and Tropical Mathematics and Problems of Mathematical Physics. Proc. Intern. Workshop, Moscow Independent University, 2007.

[7] *Marsaglia G.* – Ann. Math. Stat. 1971. V. 42, P. 1922-1929.

[8] *Brelot M.* Topologies and Boundaries in Potential Theory. Springer, Berlin, 1971.

[9] *Shahverdian A. Yu.* – Computational Methods and Function Theory. 2011. V.11, N.1, P. 71-121.

[10] *Dynkin E. B., Yushkevich A. A.* Theorems and Problems on Markov Processes. Nauka, 1967.

[11] *Kemeny J. G., Snell J. L., Knapp A. W.* Denumerable Markov Chains. Springer, N.Y. 1976.

[12] *Schuster H. G.* Deterministic Chaos. Weinheim, Berlin 1984.

[13] *Hall J. I.* Notes on Coding Theory. Michigan State University, 2010.

[14] *Graham R. L., Knuth D. E., Patashnik O.* Concrete Mathematics: a Foundation of Computer Science. Addison-Wesley, Reading, 1990.

[15] *Shub M.* In: Proc. Intern. Congress of Math. ICM-2006, P.1-22, Madrid, 2006.

[16] *Billingsley P.* Ergodic Theory and Information. Wiley, N-Y, 1965.

[17] *Martin N. F., England J. W.* Mathematical Theory of Entropy. Addison-Wesley, Massachusetts, 1981.