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Lagrangian dynamics and a Weak KAM Theorem on the d – infinite dimensional torus

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1.Introduction. Certain partial differential equations on infinite dimensional spaces can be treated via infinite dimensional dynamical systems. Consequently one needs to develop the theory of infinite dimensional dynamical systems. In the paper [2] authors define a natural Riemannian structure on the space of square integrable functions $L^2(0,1)$ on the basis of which they introduce a so called *infinite dimensional torus* T. For a certain class of Hamiltonians they were able to prove the existence of a viscosity solution to the *cell problem* on T. As an application they obtain existence of an existence of absolute action-minimizing solutions of prescribed rotation number for the one-dimensional nonlinear Vlasov system with periodic potential.

The aim of the current work is to generalize the results proven in [2] for the higher dimensional case. As a consequence one can get the existence of the solutions and asymptotics to the d-dimensional $(d \ge 1)$ nonlinear Vlasov system with periodic potential. Moreover, we prove the existence of the so called *two sided minimizers* for the cell problem over the d-infinite dimensional torus T^d.

2.Notations and definitions. I^d will denote the d-dimensional unit cube in $\mathbb{R}^d$. As usual $|\cdot|$ and $\langle\cdot,\cdot\rangle$ will respectively represent the Euclidian norm and inner product in $\mathbb{R}^d$. T^d will be the notation for the d-dimensional torus. The distance on T^d is the following:

$$|x|_{\mathbb{T}^d} = \inf_{k\in\mathbb{Z}^d} |x+k|, x \in \mathbb{T}^d.$$

L^d will be the d-dimensional Lebesgue measure on $\mathbb{R}^d$. ν_0 will be the restriction of L^d on the unit cube I^d.

We will denote by $\mathrm{L}^2\left(I^d;\mathbf{R}^d\right)$ the space of all square integrable functions defined on the unit cube $\mathrm{L}^2\left(I^d;\mathbf{R}^d\right) := \{M : \mathrm{I}^d \to \mathbb{R}^d; \int_{I^d} |M(x)|^2 \, d\nu_0 < \infty\}$. $\mathrm{L}^2_{\mathbf{Z}}\left(I^d;\mathbf{R}^d\right)$ will be the set of the square integrable functions with d-integer values: $\mathrm{L}^2_{\mathbf{Z}}\left(I^d;\mathbf{R}^d\right) := \{M \in \mathrm{L}^2\left(I^d;\mathbf{R}^d\right) : M(x) \in \mathbb{Z}^d, \forall x \in \mathrm{I}^d\}$.

$\mathrm{P}_2(\mathbb{R}^d)$ denotes the set of all Borel probability measures on $\mathbb{R}^d$ with finite second order moments: $\mathrm{P}_2(\mathbb{R}^d) := \{\mu \in \mathrm{P}(\mathbb{R}^d); \int_{\mathbb{R}^d} |x|^2 \, d\mu < \infty\}$.

If $(\mathbf{E}, |\cdot|)$ is a norm space, $L^2(0,T;\mathbf{E})$ is the set of Borel functions $M:(0,T)\to\mathbf{E}$ such that $\int_0^T |M_t|^2_{\mathbf{E}} < \infty$. We will write M_t in place of $M(t)$.

If $M:(X,\mathsf{F}) \to (Y,\mathsf{G})$ is a measurable map and μ is a measure defined on a sigma algebra F then $\nu = M\#\mu$ is the measure defined on sigma algebra G given by $\nu[C] = \mu[M^{-1}(C)]$ for all sets $C \in \mathsf{G}$.

Suppose $(\mathrm{S}, dist)$ is a complete metric space and $\sigma:(0,T)\to\mathrm{S}$. Denote by σ_t the value of σ at t. If there exists $\beta \in L^2(0,T)$ such that $dist(\sigma_t,\sigma_s) \le \int_s^t \beta(u)du$ for every $s<t$ in $(0,T)$, we say that σ is absolutely continuous. Denote by $AC^2(0,T;\mathrm{S})$ the set of all absolutely continuous paths $\sigma:(0,T)\to\mathrm{S}$.

Denote by G the set of bijections $G:\mathrm{I}^d \to \mathrm{I}^d$ such that G, G^{-1} are Borel and preserve the Lebesgue measure.

Definition 2.1. Let $U: L^2(I^d;\, \mathbb{R}^d) \to \mathbb{R}\cup\{\infty\}$. (i) We say that U is periodic if it is invariant under integer valued translations. (ii) We say that U is invariant under the action G or rearrangement invariant if $U(M\circ G) = U(M)$ for all $M \in L^2(I^d;\, \mathbb{R}^d)$ and $G \in \mathrm{G}$.

3. d – infinite **dimensional torus** T^d **and the space** $\mathcal{S}^d$. Let $\hat{\cdot}:\mathbb{R}\to\mathbb{Z}$ be the integer part function i.e. $\hat{x} = \sup\{n \in \mathbb{Z} \mid n \le x\}$ and $\pi(\cdot):\mathbb{R}\to[0,1)$ be the fractional part function i.e. $\pi(x) = x - \hat{x}$, for $x \in \mathbb{R}$. $L^2_{\mathbb{Z}}(\mathrm{I}^d)$ is a subgroup of $(L^2(\mathrm{I}^d),+)$ and thus we can consider the quotient space which will be the d-dimensional L^2-torus T^d. So we set

$$\mathrm{T}L^2(I^d;\, \mathbb{R}^d) := \mathrm{L}^2\left(I^d;\mathbf{R}^d\right)\times \mathrm{L}^2\left(I^d;\mathbf{R}^d\right), \mathrm{T}^d := \mathrm{L}^2\left(I^d;\mathbf{R}^d\right)/\mathrm{L}^2_{\mathbf{Z}}\left(I^d;\mathbf{R}^d\right).$$

Note that function π projects a function $M = (M_1,\ldots,M_d) \in \mathrm{L}^2\left(I^d;\mathbf{R}^d\right)$ onto T^d as follows

$$\pi(M(x)) = (\pi(M_1(x)), \pi(M_2(x)), \ldots, \pi(M_d(x))), x \in \mathrm{I}^d .$$

The norm on $\mathrm{L}^2\left(I^d;\mathbf{R}^d\right)$ induces distance $dist_{\mathbb{Z}}$ on T^d given by

$$dist_{\mathbb{Z}}(M_1, M_2) = \inf_{Z \in L^2_{\mathbb{Z}}(\mathrm{I}^d;\mathbf{R}^d)} \| M_1 - M_2 - Z \| . \tag{3.1}$$

Endowed with this distance T^d becomes a metric space.

Proposition3.1. $(\mathrm{T}^d, dist_{\mathbb{Z}})$ is a complete, separable metric space.

We do further factorization of the space T^d with respect to the group G. Denote by W_{T^d} the Wasserstein distance on the torus T^d. Recall that if $\mu, \nu \in P(\mathrm{T}^d)$ and $\Gamma(\mu,\nu)$ is the set of Borel measures on $\mathrm{T}^d \times \mathrm{T}^d$ which have μ and ν as marginals, then

$$W^2_{\mathrm{T}^d}(\mu,\nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathrm{T}^d \times \mathrm{T}^d} | x - y |^2_{\mathrm{T}^d} \, d\gamma(x, y) \tag{3.2}$$

We will identify $\mathrm{P}(\mathrm{T}^d)$ with $P([0,1)^d)$. The group G is a non commutative group which acts on $L^2(I^d;\mathbf{R}^d)$: $(G, M) \to M \circ G$. This is an action which preserves the norm of M. But G acts on $L^2_{\mathbb{Z}}(I^d;\mathbf{R}^d)$, it provides an action on the quotient space T^d.

The metric on T^d induces a function which is a *quasimetric* on the quotient space T^d/G i.e. for $M_1, M_2 \in L^2(I^d;\mathbf{R}^d)$ put

$$dist_{weak}(M_1, M_2) = \inf_{G \in \mathrm{G}} dist_{\mathbb{Z}}(M_1, M_2 \circ G) .$$

It is symmetric and satisfies triangle inequality because $dist_{\mathbb{Z}}$ does so. So it is an quasimetric on T^d/G. It is not a metric because there can be two functions M_1, M_2 which have 0 weak distance but their projections on T^d/G are different. We *metrize* it by gluing the functions which have 0 weak distance.

Definition 3.1. The space S_d is the factor space of the space $L^2(I)$ w.r.t. the equivalence relation $\sim: M_1 \sim M_2 \text{ } iff \text{ } dist_{weak}(M_1, M_2) = 0$. Furthermore the distance $dist_S$ is defined by the formula $dist_S([M_1],[M_2]) = dist_{weak}(M_1, M_2)$ for any equivalence classes $[M_1],[M_2] \in \mathrm{S}_d$.

Remark 3.1. One can prove that $M_1 \sim M_2 \text{ } iff \text{ } \pi(M_1)\text{й}\nu_0 = \pi(M_2)\text{й}\nu_0$.

The metric space $(\mathrm{S}_d, dist_{\mathrm{S}})$ enjoys some nice topological properties.

Theorem 3.1. The metric spaces $(\mathrm{S}_d, dist_{\mathrm{S}})$ and $(\mathrm{P}(\mathrm{T}^d), W_2)$ are isometric.

From this Theorem we immediately have:

Corollary 3.1. The space $(\mathrm{S}_d, dist_{\mathrm{S}})$ is a complete, separable, compact metric space.

We next compute the first cohomology group of T^d, and then the first equivariant cohomology group of T^d under the action of G.

Proposition 3. 2. Assume $S: L^2(I^d;\mathbf{R}^d) \to \mathbb{R}$ is Frechet differentiable and Lipschitz. (i) If dS is periodic in the sense that $d_{M+Z}S = d_M S$ for all $M \in L^2(I^d;\mathbf{R}^d)$ and $Z \in L^2_{\mathbf{Z}}(I^d;\mathbf{R}^d)$, then there exist a unique $C \in L^2(I^d;\mathbf{R}^d)$ and $U: L^2(I^d;\mathbf{R}^d) \to \mathbb{R}$ periodic such that $S(M) = U(M) + \langle C, M\rangle$. (ii) If, in addition, $M \to d_M S(M)$ is rearrangement invariant the C is a constant function and U is rearrangement ivariant.

4.Weak KAM theory on $L^2(I^d;\mathbf{R}^d)$ **.**Let $c \in \mathbb{R}^d$ and

$$L(M,N) = \frac{1}{2}\| N\|^2 - \frac{1}{2}\mathrm{W}(M),$$

together with

$$L_c(M,N) = L(M,N) - c\int_I N d\nu_0, \bar{L}(M,N) = L_c(M,-N).$$

Here, $\mathrm{W}: L^2(I^d;\mathbf{R}^d) \to \mathbb{R}$ is C^1 periodic, semiconcave and semiconvex, Lipschitz and differentiable invariant under the action of G. Define the Legendre transforms of $L(M,\cdot)$ and $L_c(M,\cdot)$:

$$H(M,N) = \frac{1}{2}\| N\|^2 + \frac{1}{2}\mathrm{W}(M),$$

and

$$H_c(M,N) = H(M,N+c), \bar{H}(M,N) = H_c(M,-N).$$

We now consider viscosity solutions to Hamilton-Jacobi equations in the infinite dimensional setting. We recall the definition of viscosity solutions.

Definition 4. 1. Let V be a real valued proper functional defined on $L^2(I^d;\mathbf{R}^d)$ with values in $\mathbb{R}\cup\{\pm\infty\}$. Let $M \in L^2(I^d;\mathbf{R}^d)$ and $\xi \in L^2(I^d;\mathbf{R}^d)$. (i) We say that ξ belongs to the subdifferential of V at M and we write $\xi \in D^- V(M)$ if $V(M+X) - V(M) \ge \langle \xi, X\rangle + o(\| X\|)$ for all $X \in L^2(I^d;\mathbf{R}^d)$. (ii) We say that ξ belongs to the superdifferential of V at M and we write $\xi \in D^+ V(M)\, if\, -\xi \in D^-(-V)(M)$.

Remark 4.1. When the sets $D^- V(M)$ and $D^+ V(M)$ are both nonempty, then they coincide and consist of a single element. That element is $\nabla V(M)$, the gradient of V at M.

We can now define the notion of viscosity solution for a general Hamilton-Jacobi equation of the type

$$F(M,\nabla U(M))=0.$$

Definition 4.2. Let $V:L^2(I^d;\mathbf{R}^d)\to\mathbb{R}$ be continuous. (i) We say that V is a viscosity subsolution for (4.1) if $F(M,\zeta)\le 0$ for all $M\in L^2(I^d;\mathbf{R}^d)$ and all $\zeta\in D^+V(M)$. (ii) We say that V is a viscosity supersolution for (4.1) if $F(M,\zeta)\ge 0$ for all $M\in L^2(I^d;\mathbf{R}^d)$ and all $\zeta\in D^-V(M)$. (iii) We say that V is a viscosity solution for (4.1) if V is both a subsolution and a supersolution for (4.1).

4.1. A preliminary stationary Hamilton-Jacobi equation. Define an action

$$\mathrm{A}_\varepsilon(x):=\int_0^\infty e^{-\varepsilon t}\tilde{L}(x,\dot{x})dt,$$

which is well defined for $x\in AC^2_{loc}((0,\infty);L^2(I^d;\mathbf{R}^d))$, since $\tilde{L}$ is bounded by below by $-c^2/2$. We do not display its dependence on c to keep notation simpler. Set

$$V_\varepsilon(M):=\inf_x\{\mathrm{A}_\varepsilon(x):\ x\in AC^2_{loc}((0,\infty);\ L^2(I^d;\mathbf{R}^d))\} \tag{4.2}$$

Since $\tilde{L}$ is invariant under the action of G, so is V_ε. The fact that $\tilde{L}(\cdot,N)$ is periodic ensures that V_ε is periodic.

In [2] it was proven that in the one dimensional case, when M is monotone nondecreasing (4.2) admits a minimizer. This minimizer is in $H^2_{loc}((0,\infty);L^2(I^d;\mathbf{R}^d))$ and satisfies the Euler-Lagrange equation. We have shown that for every $d\ge 1$ the minimizers exist on an everywhere dense G_δ subspace of the space $L^2(I^d;\mathbf{R}^d)$.

One the most important features of the value function V_ε is the following:

Theorem 4.1. V_ε is a Lipschitz function and a viscosity solution of the equation

$$\varepsilon V_\varepsilon(M)+\tilde{H}(M,\nabla_{L^2}V_\varepsilon(M))=0. \tag{4.3}$$

Furthermore we have that:

Proposition 4.1. The superdifferential D^-V_ε is not empty at every point $M\in L^2(I^d;\mathbf{R}^d)$. Moreover V_ε is differentiable on an everywhere dense G_δ set.

Main result concerning variational problem 4.2 is the following theorem:

Theorem 4.2. Problem (4.2) admits a unique minimizer for any differentiability point $M\in L^2(I^d;\mathbf{R}^d)$ of the value function V_ε.

4.2.The cell problem. Consider function $U_\varepsilon := V_\varepsilon - \inf V_\varepsilon$, where V_ε is the value function defined in the previous section. One can show that:

Proposition 4.2. (i) The function U_ε is Lipschitz and $U_\varepsilon(M) = U_\varepsilon(\bar{M})$ whenever $[M] = [\bar{M}]$.

(ii) Every subfamily of $\{U_\varepsilon\}_{\varepsilon\in(0,1)}$ admits a subsequence converging uniformly to some U which is κ-Lipschitz. Every subfamily of $\{\varepsilon V_\varepsilon\}_{\varepsilon\in(0,1)}$ admits a subsequence converging uniformly to a constant depending on c which we denote $-\bar{H}(c)$.

The limit function U enjoys a nice variational representation.

Proposition 4.3. For any $T > 0$ and any $M \in L^2(I^d; \mathbf{R}^d)$

$$U(M) = \inf\{\int_0^T \tilde{L}(x(s), \dot{x}(s)) + \bar{H}(c)ds + U(x(T));\ x(0) = M,\ x \in AC^2((0,T); L^2(I^d; \mathbf{R}^d))\} \tag{4.4}$$

Theorem 4.3. U is a viscosity solution to the equation

$$H(M, c + \nabla_{L^2} U) = \bar{H}(c). \tag{4.5}$$

Additionally for every differentiability point $M \in L^2(I^d; \mathbf{R}^d)$ of the function U there exists a unique trajectory $x \in C^2([0,\infty); L^2(I^d; \mathbf{R}^d))$ which is a minimizer for the problem (4.4) for all times $T > 0$ and which satisfies Euler-Lagrange equation

$$\ddot{x} = -\frac{1}{2}\nabla_{L^2} \mathrm{W}(x), \tag{4.6}$$

with $x(0) = M$ and $\dot{x}(0) = -(c + \nabla U(M))$. Furthermore, x minimizes the action $\widetilde{\mathrm{A}}_T(y) = \int_0^T \left(\tilde{L}(y(s), \dot{y}(s)) + \bar{H}(c) \right) ds$ over all trajectories $y \in AC^2((0,T); L^2(I^d; \mathbf{R}^d))$ with endpoints $y(0) = x(0)$ and $y(T) = x(T)$.

We also show the existence of so called two sided minimizers of the action $\widetilde{\mathrm{A}}_{t_1}^{t_2}(y) = \int_{t_1}^{t_2} \left(\tilde{L}(y(s), \dot{y}(s)) + \bar{H}(c) \right) ds$.

Theorem 4.4. There exist points $(x, p) \in \mathrm{T}^* L^2(I^d; \mathbf{R}^d)$ s.t. there exists a unique trajectory $x \in C^2(\mathbb{R}; L^2(I^d; \mathbf{R}^d))$ which satisfies the Euler-Lagrange equation (4.6), passes through $(x, p): x(0) = x,\ p(0) = p$, and minimizes the action $\widetilde{\mathrm{A}}_{t_1}^{t_2}$ for all times $t_1 < t_2$.

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Lagrangian dynamics and a Weak KAM theorem on the d – infinite dimensional torus

The space $L^2(0,1)$ has a natural Riemannian structure of the basis of which in their recent work W. Gangbo and A. Tudorascu introduced an infinite dimensional torus T. For a certain classes of Hamiltonians they prove an existence of a viscosity solution to the cell problem on T. In the current work we generalize obtained results for the so called higher dimensional case, where we start with the space $L^2\left((0,1)^d;\mathbb{R}^d\right)$ and introduce the d – infinite dimensional tours T^d.

Լ. Նուրբեկյան, Դ. Գոմես

Լագրանժյան դինամիկա և թույլ ԿԱՄ թեորեմ d – անվերջ չափանի տորի վրա

$L^2(0,1)$ տարածությունն օժտված է բնական Ռիմանյան կառուցվածքով: Հիմնվելով այդ կառուցվածքի վրա, Վ. Գանգբոն և Ա. Տուդորասկուն ներմուծել են *անվերջ չափանի* T տորի գաղափարը: Համիլտոնյանների որոշակի դասի համար նրանք ապացուցել են *տարրական* Համիլտոն- Յակոբի հավասարման թույլ լուծման գոյությունը T-ի վրա: Սույն աշխատանքում մենք ընդհանրացնում ենք վերը նշված արդյունքը բազմաչափ դեպքի համար:

Л. Нурбекян, Д. Гомес

Лагранжева динамика и слабая КАМ теорема на d -бесконечномерном торе

Пространство $L^2(0,1)$ имеет естественную Риманову структуру. На основании этой структуры В. Гангбо и А. Тудораску в одной из недавних работ ввели понятие *бесконечномерного* тора T. Впоследствии им удалось доказать существование вязкостного решения элементарной задачи Гамильтона – Якоби. Эти результаты нами обобщены для так называемого d -бесконечномерного торуса T^d, полученного из пространства $L^2\left((0,1)^d;\mathbb{R}^d\right)$ аналогичным образом.

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