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General Classification of Normally Flat *Ric*-Semisymmetric Submanifolds

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Key words: Ric – semisymmetric manifolds, Einstein submanifolds, semi-Einstein submanifolds, interlacing products of submanifolds.

1. Riemannian Ric – semisymmetric manifolds are characterized by the condition of semi-parallelism of Ricci tensor R_1 ($R(X,Y)R_1 = 0$) and have been the subject of investigation over the last forty years (see [1, 2] and the literature cited there). The interest towards them stems from the fact that they are the generalizations of Riemannian symmetric, semisymmetric and Einstein manifolds. Some particular classes of Ric-semisymmetric manifolds and submanifolds were investigated in [1-12].

In this article the general classification of normally flat Ric-semisymmetric submanifolds in Euclidean spaces is presented.

2. Let M be a Riemannian manifold with Riemannian connection ∇ and curvature tensor R. It is known that curvature operators $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ act as differentiations of tensor algebra on M. For example,

$$(R(X,Y)R_1)Z = R(X,Y)R_1(Z) - R_1(R(X,Y)Z),$$

where X, Y, Z are the arbitrary tangent vector fields on M. It is also known that at every point $x \in M$ Ricci tensor R_1 acts as a symmetric endomorphism of the tangent space $T_x(M)$. If $R_1 = 0$, then the Riemannian manifold is said to be ricci-flat. If $R_1 = \lambda I$, where $\lambda = const$ and I is the identity transformation, then the manifold M is called Einsteinian. If $R(X,Y)R_1 = 0$ for any X, Y (which is equivalent to the condition $R(X,Y) \cdot R_1 = R_1 \cdot R(X,Y)$), then the Riemannian manifold is called $Ric-$semisymmetric. It is proved by the author, that the smooth Riemannian manifold M satisfies the condition $R(X,Y)R_1 = 0$ if and only if it is an open part of the direct product of two-dimensional, Einstein and semi-Einstein submanifolds [13].

3. Let M be a Riemannian manifold and $x \in M$ be an arbitrary point. The subspace $T_x^{(0)} = \{X \in T_x(M);\ R(X,Y) = 0\, \forall\, Y \in T_x(M)\}$ of the tangent space $T_x(M)$ is called nullity space at point x, and its dimension $\mu_x = \dim T_x^{(0)}$ is said to be the index of nullity at x. The distribution $T^{(0)}$ (the nullity distribution) is integrable and totally geodesic, and its integral manifold is locally Euclidean in the induced metric [14]. The space $T_x^{(0)}$ lies in the subspace of eigenvectors of the tensor R_1, corresponding to the zero eigenvalue. The orthogonal complement $T_x^{(1)}$ of the space $T_x^{(0)}$ in $T_x(M)$ with respect to the Riemannian metric on M is called conullity space at x, and its dimension is called conullity index at this point. The space $T_x^{(1)}$ is invariant under the operators $R(X,Y)$ and the tensor R_1. Consequently, at every point $x \in M$ the Ricci tensor R_1 has two invariant subspaces $T_x^{(0)}$, $T_x^{(1)}$ and we have the direct sum decomposition $T_x(M) = T_x^{(0)} + T_x^{(1)}$ (see [13] for details). The Riemannian manifold M with non-zero nullity index is called semi-Einstein, if Ricci tensor R_1 has only one non-zero eigenvalue on $T_x^{(1)}$. Examples of semi-Einstein manifolds are the Riemannian manifolds of conullity 2 [1], as well as cones over Einstein manifolds with negative Einstein constants [10].

4. Let M be the submanifold of n-dimensional Euclidean space E_n. By R, $R^{\perp}$, α_2, A_{ξ} we will denote, respectively, the curvature tensor of Riemannian connection of induced metric on M, the curvature tensor of normal connection, the second fundamental form and the second fundamental tensor with respect to normal vector field ξ (см. [15]). If $R=0$ the submanifold M is called locally Euclidean, and if $R^{\perp}=0$, it is called normally flat. Ricci tensor R_1 for M is defined in a standard way. If $\alpha_2=0$, then the submanifold M is called totally geodesic.

5. An isometric immersion $M \to E_n$ is said to be the product of immersions $M_{\varphi} \to E_{n_{\varphi}}$, if $M = M_1 \times \cdots \times M_r$, $E_n = E_{n_1} \times \cdots \times E_{n_r}$ and any two subspaces $E_{n_{\varphi}}$ and $E_{n_{\psi}}$ $(\varphi \neq \psi)$ are totally orthogonal in E_n. In this case we say that M is the direct product of the submanifolds $M_1, \cdots, M_r$, or it is reducible (as submanifold). If $M = M_1 \times \cdots \times M_r$ in E_n is irreducible as submanifold, then we will say, that M is the interlacing product of the submanifolds $M_1, \cdots, M_r$. Considering reducibility of submanifolds in E_n we will relay on the following result.

Theorem 1. *Let U be a domain of a submanifold M of E_n and $\Delta_1, \cdots, \Delta_r$ pairwise totally orthogonal integrable distributions in U $\left(\Delta_1(x) + \cdots + \Delta_r(x) = T_x(U)\ \forall x \in U\right)$ with integral manifolds $M_1, \cdots, M_r$ respectively. Then the domain U is the product of the submanifolds $M_1, \cdots, M_r$ if and only if $\Delta_1, \cdots, \Delta_r$ are parallel with respect to the Riemannian connection ∇ on M and are conjugate with respect to the second fundamental form α_2 $\left(\alpha_2(X,Y)=0\ \forall X \in \Delta_{\varphi}(x),\ \forall Y \in \Delta_{\psi}(x),\ \varphi \neq \psi\right)$.*

The necessity of the conditions of Theorem 1 is easily proved (see, for example, [2]), the sufficiency of these conditions is the subject of Moore's basic lemma [16].

6. In our investigations V- and Z- decompositions of the tangent space of a Riemannian manifold by Z. Szabó will be used [5]. The construction of these decompositions is as follows. Let M be a Riemannian manifold and let $x \in M$ be a fixed point. In the linear space of skew-symmetric linear operators $T_x(M) \to T_x(M)$ let us consider the linear subspace h_x, spanned by elements $R_x(X,Y)$, where $X, Y \in T_x(M)$, that is to say, $h_x = span\, R_x(X,Y)$. For arbitrary two elements $R_x(X,Y)$ and $R_x(Z,W)$ from h_x let us define the commutator according to the formula $[R_x(X,Y), R_x(Z,W)] = R_x(X,Y) \cdot R_x(Z,W) - R_x(Z,W) \cdot R_x(X,Y)$. Let $\overline{h_x}$ be Lie's algebra, generated by the set h_x with respect to this operation, and let P_x be the connected subgroup of the isometry group in $T_x(M)$, defined by $\bar{h}_x$. This group is called primitive holonomy group at x. Let $T_x(M)=V_x^{(0)}+V_x^{(1)}+\cdots+V_x^{(t)}$ be the irreducible decomposition of the space $T_x(M)$ with respect to P_x. The subspaces $V_x^{(\rho)}$ are invariant with respect to the action of P_x and pairwise totally orthogonal. Moreover, P_x acts trivially on $V_x^{(0)}$, and irreducibly on $V_x^{(\rho)}, \rho>0$. This decomposition is called $V-$decomposition of the space $T_x(M)$. It is easy to show, that $V_x^{(0)}$ coincides with the nullity space $T_x^{(0)}$.

Theorem 2 (Z. Szabó [5]). *There exists on a Riemannian C^∞ manifold M an open dense subset G in which the subspaces $V_x^{(\rho)}$ have constatnt dimensions, $V-$decomposition is unique up to the order of the terms, and the corresponding distributions $V^{(\rho)}$ have the following properties on G:*

$$\nabla_{V^{(0)}}V^{(0)}\subseteq V^{(0)}, \qquad \nabla_{V^{(0)}}V^{(\rho)}\subseteq V^{(\rho)}, \qquad \nabla_{V^{(\rho)}}V^{(0)}\subseteq V^{(0)}+V^{(\rho)},$$

$$\nabla_{V^{(\rho)}}V^{(\rho)}\subseteq V^{(0)}+V^{(\rho)}, \qquad \nabla_{V^{(\rho)}}V^{(\tau)}\subseteq V^{(\tau)}, \qquad \rho\neq\tau, \quad \rho,\tau\neq 0,$$

where the notation $\nabla_{V^{(\rho)}}V^{(\tau)}\subseteq V^{(\sigma)}$ indicates, that for each $X\in V^{(\rho)}$ and each $Y\in V^{(\tau)}$ the vector $(\nabla_X Y)_x$ lies in $V_x^{(\sigma)}$.

From above inclusions it follows that the distributions $V^{(\rho)}$, generally speaking, are not integrable and not parallel on M (though parallelism of some of them is not excluded). However they can always be extended (in the sense of dimension) to parallel (and consequently integrable) distributions by Z. Szabó's method [5], the essence of which is as follows. Let $Z_x^{(\rho)}$ – be a subspace in $T_x(M)$, spanned by the vectors

$$X_{1|x},\ \nabla_{X_1}X_{2|x},\ \nabla_{X_1}\nabla_{X_2}X_{3|x},\cdots,\ \nabla_{X_1}\cdots\nabla_{X_k}X_{k+1|x},\cdots,$$

where all the X_k belong to $V^{(\rho)}$, $\rho>0$. By definition we put $Z_x^{(0)}=\left(Z_x^{(1)}+\cdots+Z_x^{(t)}\right)^{\perp}$, where $(\ )^{\perp}$ denotes the orthogonal complement in $T_x(M)$. It is easy to see, that $Z_x^{(0)}\subseteq V_x^{(0)}$, $V_x^{(\rho)}\subseteq Z_x^{(\rho)}$, $\rho>0$. It follows from the inclusions of Theorem 2, that the extension of the subspace $V_x^{(\rho)}$, $\rho>0$, to $Z_x^{(\rho)}$ proceeds only at the expense of subspace $V_x^{(0)}$.

Theorem 3 (Z.Szabó [5]). *The subspaces* $Z_x^{(\rho)}$, $x \in G$, *are pairwise totally orthogonal, and there exists an open dense subset* $\overline{G} \subset G \subset M$, *on which the* $Z_x^{(\rho)}$ *have constant dimensions and the corresponding distributions* $Z^{(\rho)}$ *are parallel in the Riemannian connection on* M .

The decomposition $T_x(M) = Z_x^{(0)} + Z_x^{(1)} + \cdots + Z_x^{(t)}$ is called $Z-$decomposition of the space $T_x(M)$.

7. Let us now turn to the basic objective of the article − the proof of the following theorem.

Theorem 4. *A normally flat submanifold* M *in Euclidean space* E_n *is* Ric*-semisymmetric if and only if it is an open part of one of the following submanifolds:*

(1) normally flat two-dimensional submanifold,

(2) normally flat Einstein submanifold (in particular Ricci-flat or locally Euclidean),

(3) normally flat semi-Einstein submanifold,

(4) normally flat interlacing product of semi-Einstein submanifolds and locally Euclidean submanifold (may be of zero dimension),

(5) direct product of the above enumerated classes of submanifolds.

Proof. Let $\Delta_1(x), \cdots, \Delta_r(x)$ be the subspaces of eigenvectors (eigenspaces) of tensor R_1 in the tangent space $T_x(M)$, and $\Delta_1, \cdots, \Delta_r$ indicate the corresponding distributions (eigen-distributions). As in the case of normally flat connection the tensor R_1 commutes with all the tensors A_ξ (see [17]), then the subspaces $\Delta_1(x), \cdots, \Delta_r(x)$ are conjugated with respect to the second fundamental form α_2. It is known [13], that $R(X,Y)Z \in \Delta_\varphi(x)$ for any $Z \in \Delta_\varphi(x)$ and any X, Y, $R(X,Y) = 0$ for any $X \in \Delta_\varphi(x)$ and any $Y \in \Delta_\psi(x)$, $\varphi \neq \psi$, $R(X,Y)Z = 0$ for any $X, Y \in \Delta_\varphi(x)$ and any $Z \in \Delta_\psi(x)$, $\varphi \neq \psi$. The latter means that the endomorphisms $R(X,Y)$, $X, Y \in \Delta_\varphi(x)$ act trivially on $\Delta_\psi(x)$, $\psi \neq \varphi$. Similarly, as mentioned above, we have a linear subspace $h_x^{(\varphi)} = span R(X,Y)$, $(X, Y \in \Delta_\varphi(x))$ in the linear space of all skew-symmetric endomorphisms $\Delta_\varphi(x) \to \Delta_\varphi(x)$. The subspace $h_x^{(\varphi)}$ turns into Lie's algebra with respect to the bracket operation defined above and it is the subalgebra of the h_x algebra. Consequently, the primitive holonomy group P_x is reducible and $P_x = P_x^{(1)} \times P_x^{(2)} \times \cdots \times P_x^{(r)}$ decomposition takes place, where $P_x^{(\varphi)}$ is the subgroup with Lie's algebra $h_x^{(\varphi)}$. The subgroup $P_x^{(\varphi)}$ acts trivially on $\Delta_\psi(x)$, $\psi \neq \varphi$, and, generally speaking, it can be reducible on $\Delta_\varphi(x)$. Let $\Delta_1(x)$ be eigensubspaces of tensor R_1, corresponding to zero eigenvalue, and let

$$\Delta_1(x) = V_x^{(0)} + V_x^{(1,1)} + \cdots + V_x^{(1,s_1)}, \qquad \Delta_\varphi(x) = V_x^{(\varphi,1)} + \cdots + V_x^{(\varphi,s_\varphi)},\ \varphi > 1,$$

be irreducible decompositions of $\Delta_1(x)$ and $\Delta_\varphi(x)$, $\varphi > 1$, with respect to the subgroups $P_x^{(1)}$ and $P_x^{(\varphi)}$, $\varphi > 1$, respectively. The first decomposition is based on fact that $V_x^{(0)} \subset \Delta_1(x)$. Subspaces $V_x^{(\varphi,\, l_\varphi)}$ $(l_\varphi = 1, \cdots, s_\varphi)$ are invariant with respect to actions of $P_x^{(\varphi)}$ (and consequently to those of P_x) and pairwise totally orthogonal. Moreover $P_x^{(\varphi)}$ (and consequently also P_x) acts trivially on $V_x^{(0)}$ and irreducibly on $V_x^{(\varphi,\, l_\varphi)}$. Consequently, the decomposition

$$T_x(M) = V_x^{(0)} + V_x^{(1,1)} + \cdots + V_x^{(1,s_1)} + \cdots + V_x^{(r,1)} + \cdots + V_x^{(r,s_r)} \tag{1}$$

is $V-$decomposition of the space $T_x(M)$. Let us note that $\dim \Delta_\varphi(x) \geq 2$ if $\varphi \geq 2$, because if $\dim \Delta_\varphi(x) = 1$ for some φ, then $\Delta_\varphi(x) \subset V_x^{(0)}$.

Let the nullity index $\mu = 0$, that is $V_x^{(0)}$ is trivial. In this case the system of inclusions in Theorem 2 reduces to the following: $\nabla_{V^{(\rho)}} V^{(\rho)} \subseteq V^{(\rho)}$, $\quad \nabla_{V^{(\rho)}} V^{(\tau)} \subseteq V^{(\tau)}$, $\quad \rho \neq \tau$. It follows from here, that the distributions $V^{(\rho)}$ are parallel. Based on this, we come to the conclusion, that the distributions $V^{(\varphi,\, l_\varphi)}$ are parallel. Then each distribution Δ_φ, being the sum of such distributions, is also parallel. As $\Delta_\varphi(x)$ is conjugated with respect to the second fundamental form α_2, then, according to Theorem 1, the submanifold M is (locally) the direct product of integral manifolds of the distributions Δ_φ. As on each subspace $\Delta_\varphi(x)$ the tensor R_1 has only one eigenvalue, then the integral manifold of the distribution Δ_φ is either two-dimensional, or Einstein with zero nullity index in both cases. It only remains to note, that the direct product of submanifolds is normally flat if and only if when each factor-submanifold is normally flat (see, for example, [2], [10], [15]).

Let $\mu \geq 1$, that is $V_x^{(0)}$ not be a null space. In this case, according to the above constructed $V-$decomposition (1), we can construct the corresponding $Z-$decomposition

$$T_x(M) = Z_x^{(0)} + Z_x^{(1,1)} + \cdots + Z_x^{(1,s_1)} + \cdots + Z_x^{(r,1)} + \cdots + Z_x^{(r,s_r)}, \tag{2}$$

where $Z_x^{(0)} \subseteq V_x^{(0)}$, $V_x^{(\varphi,l_\varphi)} \subseteq Z_x^{(\varphi,l_\varphi)}$, and the distributions $Z^{(0)}$, $Z^{(\varphi,l_\varphi)}$ are parallel in the Riemannian connection on M (Theorem 3). As while constructing Z decompositions (2) the possible extensions (in the sense of dimension) of subspaces $V_x^{(\varphi,l_\varphi)} \subseteq \Delta_\varphi(x)$ take place only due to the subspace $V_x^{(0)}$ (it follows from the inclusions in Theorem 2), then the possible extensions of subspaces $\Delta_\varphi(x)$, $\varphi > 1$, also take place only due to the subspace $V_x^{(0)}$. Let us denote these extensions by $\widetilde{\Delta}_\varphi(x)$, that is to say $\widetilde{\Delta}_\varphi(x) = Z_x^{(\varphi,1)} + ... + Z_x^{(\varphi,s_\varphi)}$ if $\varphi \geq 2$. Then $T_x(M) = \widetilde{\Delta}_1(x) + \widetilde{\Delta}_2(x) + \cdots + \widetilde{\Delta}_r(x)$, where $\widetilde{\Delta}_1(x) = Z_x^{(0)} + Z_x^{(1,1)} + \cdots + Z_x^{(1,s_1)}$, and $\widetilde{\Delta}_1(x) \subseteq \Delta_1(x)$. Distributions $\widetilde{\Delta}_1, \cdots, \widetilde{\Delta}_r$, being the sums of parallel distributions, are also parallel on submanifold M and, consequently, integrable. Let us note, that on $\widetilde{\Delta}_1(x)$ the Ricci tensor has only zero eigenvalue. If any $\widetilde{\Delta}_\varphi(x)$, $\varphi \geq 2$, coincides with $\Delta_\varphi(x)$, then on $\widetilde{\Delta}_\varphi(x)$ the Ricci tensor has only one non-zero eigenvalue, and, consequently, the integral manifold of distribution $\widetilde{\Delta}_\varphi$ is either two-dimensional or Einstein. And if $\dim \widetilde{\Delta}_\varphi(x) > \dim \Delta_\varphi(x)$, then the Ricci tensor has only two eigenvalues on $\widetilde{\Delta}_\varphi(x)$: non-zero on $\Delta_\varphi(x)$ and zero on the orthogonal complement $\Delta_\varphi(x)$ in $\widetilde{\Delta}_\varphi(x)$. Consequently, the integral manifold of distribution $\widetilde{\Delta}_\varphi$ is semi-Einstein.

Let us consider some possible cases.

A. Let the subspaces $\widetilde{\Delta}_1(x), \cdots, \widetilde{\Delta}_r(x)$ be conjugate with respect to the second fundamental form α_2. Then, according to Theorem 1, M is locally the direct product of two-dimensional, Einstein and semi-Einstein submanifolds.

B. Let the set $\widetilde{\Delta}_1(x),\cdots,\widetilde{\Delta}_r(x)$ cannot be decomposed into two groupes such that each subspace of one groupe is conjugated with each subspace of the second group or with their direct sum with respect to α_2. Then $\widetilde{\Delta}_\varphi(x)$ cannot coincide with $\Delta_\varphi(x)$ for any $\varphi = 2,...,r$. Indeed, if any $\widetilde{\Delta}_\varphi(x)$ coincides with $\Delta_\varphi(x)$, then it will be conjugate with all its orthogonal complement in $T_x(M)$ due to analogical property $\Delta_\varphi(x)$, which contradicts the assumption. In this case M is irreducible as a submanifold in E_n. Internally it is the direct product of Ricci-flat subanifold and semi-Einstein submanifolds. However, resulting from the fact that $\widetilde{\Delta}_\varphi(x)$ are not conjugated with respect to the form α_2, it follows, that the first normal spaces of integral manifolds of distributions $\widetilde{\Delta}_\varphi$ interlace. Consequently, M is normally flat interlacing product of Ricci-flat submanifold (probably of zero dimension) and semi-Einstein submanifolds.

C. Let us consider now the most general situation. Let from the set of subspaces $\widetilde{\Delta}_\varphi(x)$ $(\varphi \geq 2)$ be separated all those subspaces $\widetilde{\Delta}_\psi(x)$, which are not extensions, that is to say $\widetilde{\Delta}_\psi(x)$ coincides with $\Delta_\psi(x)$. They will conjugate among one other, as well as with the orthogonal complement of their direct sum. From the remained set of subspaces $\widetilde{\Delta}_\varphi(x)$ $(\varphi \neq 1)$ let us also separate all those subspaces $\widetilde{\Delta}_\chi(x)$, which are the extensions of the corresponding subspaces $\Delta_\chi(x)$ and conjugate both among one another, and with the direct sum of the remaining subspaces $\widetilde{\Delta}_\varphi(x)$. At last, let us consider, that the whole remaining set of subspaces $\widetilde{\Delta}_\varphi(x)$ is decomposed into disjoint sets so, that all the subspaces of each set conjugate with each subspace of any of the other sets or with their direct sum. In this case M will look like a direct product $N_0 \times N_1 \times ... \times N_k$, where N_0 is the direct product of normally flat two-dimensional, Einstein and semi-Einstein submanifolds, and $N_1,\cdots,N_k$ are the normally flat interlacing products of various sets of semi-Einstein submanifolds and Ricci-flat submanifold (may be of zero dimension) in one of the sets. The necessity of conditions of the theorem is proved.

The sufficiency of the conditions of the theorem stems from the fact, that normally flat, two-dimensional, Einstein and semi-Einstein submanifolds, as well as the above presented normally flat interlacing products of semi-Einstein submanifolds and Ricci-flat submanifold are $Ric-$semisymmetric, and the condition of $Ric-$semisymmetry is the inner and multiplicative property (see [13]). The theorem is proved.

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General Classification of a Normally Flat Ric- Semisymmetric Submanifolds

It has been proved that a normally flat submanifold M in Euclidean space E_n satisfies the condition $R(X,Y)Ricci=0$ if and only if it is the open part of one of the following submanifolds: (1) normally flat two-dimensional submanifold, (2) normally flat Einstein submanifold (in particular Ricci-flat or locally Euclidean), (3) normally flat semi- Einstein submanifold, (4) normally flat interlacing product of semi-Einstein submanifolds and locally Euclidean submanifold (may be of zero dimension), (5) direct product of the above enumerated classes of submanifolds.

Վ. Ա. Միրզոյան

Նորմալ հարթ $Ric-$ կիսասիմետրիկ ենթաբազմաձևությունների ընդհանուր դասակարգումը

Ապացուցված է, որ E_n էվկլիդեսյան տարածությունում նորմալ հարթ M ենթաբազմաձևությունը բավարարում է $R(X,Y)Ricci=0$ պայմանին այն, և միայն այն դեպքում, երբ նա հանդիսանում է հետևյալ ենթաբազմաձևություններից մեկի բաց մաս՝ (1) նորմալ հարթ երկչափ ենթաբազմաձևության, (2) նորմալ հարթ էյնշտեյնյան (մասնավորապես րիչչի-հարթ, լոկալ էվկլիդեսյան) ենթաբազմաձևության, (3) նորմալ հարթ կիսաէյնշտեյնյան ենթաբազմաձևության,

(4) կիսաէյնշտեյնյան ենթաբազմաձևությունների և ռիչչի-հարթ ենթաբազմաձևության (հնարավոր է զրո չափի) նորմալ հարթ միահյուսվող արտադրյալի, (5) վերը թվարկած ենթաբազմաձևությունների դասերի ուղիղ արտադրյալի:

В. А. Мирзоян

Общая классификация нормально плоских

Ric - полусимметрических подмногообразий

Доказано, что в евклидовом пространстве E_n нормально плоское подмногообразие M удовлетворяет условию $R(X,Y)Ricci = 0$ тогда и только тогда, когда оно является открытой частью одного из следующих подмногообразий: (1) нормально плоского двумерного подмногообразия, (2) нормально плоского эйнштейнова (в частности, риччи-плоского, локально евклидова) подмногообразия, (3) нормально плоского полуэйнштейнова подмногообразия, (4) нормально плоского сплетающегося произведения полуэйнштейновых подмногообразий и риччи-плоского подмногообразия (возможно размерности ноль), (5) прямого произведения перечисленных выше классов подмногообразий.

References

1. *Boeckx E., Kowalski O., Vanhecke L.* Riemannian manifolds of conullity two. Singapore – River Edge: World Sci. Publ. NJ. 1996.
2. *Lumiste Ü.* Semiparallel submanifolds in space forms. New York: Springer. 2009.
3. *Ryan P.* - Tôhoku Math. J. 1969.V. 21. N2. P.363-388.
4. *Ferus D.* - Math. Ann. 1980. V. 247.N1. P.81-93.
5. *Szabo Z.I.* - J. Differential Geom. 1982. V. 17. N4. P. 531-582.
6. *Szabo Z.I.* – Acta Sci. Math. 1984. V.47. N3-4. P.321-348.
7. *Dillen F., Nölker S.* - J. Reine Angew. Math. 1993. V.435. P.33-63.
8. *Kowalski O.* - Czech. Math. J. 1996. V.46(121). P. 427-474.
9. *Mirzoyan V.A.* - Mat. Sb. 2000. V. 191. N9. P. 65-80.
10. *Mirzoyan V.A.* - Mat. Sb. 2006. V. 197. N7. P. 47-76.
11. *Mirzoyan V.A.* - Mat. Sb. 2008. V. 199. N3. P. 69-94.

12. *Mirzoyan V.A.* - Reports of NAS RA. 2010. V. 110. N 4. P. 340-347.

13. *Mirzoyan V.A.* - Izv. Vyssh. Uchebn. Zaved. Mat. 1992. N6. P. 80-89.

14. *Chern S.S., Kuiper N.* - Ann. of. Math. 1952. V. 56. N3. P.422-430.

15. *Chen B.-У.* Geometry of submanifolds. New York: Marcel Dekker. 1973.

16. *Moor J.D.* - J. Differential Geom. 1971. V. 5. P. 159-168.

17. *Mirzoyan V.A.* - Izv. Vyssh. Uchebn. Zaved. Mat. 1993. N9. P. 22-27.