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Avalanches and Memory in Rotator Networks

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Key words: *dynamical networks, avalanches, memory modeling, entropy, capacities*

1. Introduction. In this Section we describe our statement and the obtained results. This paper proposes a mechanism for operating of a memory in some dynamical networks. We consider the rotator networks (RN) that were introduced in [1]-[3] and characterized as a rotator model of self-organized criticality (SOC). The concept of SOC was outlined in statistical physics by Bak, Tang, and Wiesenfeld [4] and concerns the systems consisting of a large number of interacting threshold microsystems. The notion of an avalanche — an event of simultaneous attaining some threshold states by a number of microsystems — is the basic one in this theory.

Later, their method in economics, biology, and neuroscience was used. For instance, the Herz-Hopfield neural network of integrate-and-fire neurons in [5] was suggested. In such neurons the membrane voltage of a neuron can be (roughly) treated as a rotator: the voltage increases, when reaching a threshold it drops to a level close to zero, neuron's firing occurs and the cycle is started again. Similar networks are considered in [6] where the presence of unstable attractors (Milnor attractors), which could be applicable to studying the memory, is claimed. Some statements related to both avalanches and memory, in [7] are considered.

An RN is a collection of rotators assigned on a finite set X — with every $x \in X$ a phase-shifting rotator (PSR), is associated. A PSR is a semi-classical mechanical system [8] consisting of a particle P rotating on a circle C on which a threshold mark $\rho \in C$, is assigned. There is a topological structure on X (so-called network topology) and PSR from neighbor sites may interact via exchange of instantaneous δ-kicks of a given intensity α (coupling constant).

Our statement originated from the brain theory: how the brain, whose (neural) micro-dynamics is recognized as a chaotic one, enables to store unchanged the patterns in his memory? The term avalanche memory that we use can be understood in two contexts — it reflects the ability of a network to identify (restore) a given pattern (a pattern is a union of connected sets on X, a cluster) by means of avalanches as well as denotes the actual collection of different clusters which can be restored (retrieved) as a result of such identification. The clusters on X are considered as the only information-related units in the network.

The content of this paper is the following. In Section the definitions of some notions are given. In Section our main results are presented — we are interested in a question how by means of avalanches a given cluster K on X can be identified. We claim that this task can be resolved by including into a network of independent rotators some arbitrarily weak interaction between the sites of K. The recognition criterion that we use is defined by some modified Shannon entropy and then is expressed in terms of some discrete capacity C_K.

As in [2, 3] we in fact deal with a version of the Turing-Smale statement on oscillatory biological networks [9] but where the information-related notions are now involved. We note that being characterized as a SOC model the RN can also be classified to phase models of coupled oscillators investigated by Winfree and then by Kuramoto (see, e.g., [10]), while the formulation of our main Theorem 4 may remind us some results from potential theory (e.g., Carleson theorem on removable singularities, [11]).

2. Some Definitions. 2.1. A capacity for clusters. Let X be an abstract finite set and $2^X = \{S : S \subseteq X\}$ be the collection of all of its subsets. The entries of X are referred to as sites or vertices; the $|S|$ denotes the cardinality (the number of entries) of S and it is assumed that $|X| \geq 2$. Let $\sigma : X \to 2^X$ be a map assigning to every $x \in X$ a non-empty subset $\sigma(x) \subseteq X$, $x \notin \sigma(x)$. The σ assigns a topology or a neighborhood relation on X: the entries of $\sigma(x)$ are understood as the neighbors for x. A set $S \subseteq X$ is called a connected set if for every pair of different vertices $u, v \in S$ there exist some vertices $u_0, \ldots, u_n$ of S such that $u_0 = u$, $u_n = v$, and u_{i-1} and u_i ($1 \leq i \leq n$) are the neighbors with respect to σ. Every set can be represented as union of pairwise disjoint connected subsets (connected components). We assume the reflexivity of σ: $x' \in \sigma(x)$ whenever $x \in \sigma(x')$; then the clusters can be defined: a set $S \subseteq X$ is called a cluster on X, if each of its connected components contains at least two different vertices.

Definition 1 *Let $S \subseteq X$ be a cluster and (S_0, S_1) be its partition into two subsets S_0 and S_1: $S_0 \cup S_1 = S$, $S_0 \cap S_1 = \emptyset$, $S_0 \neq \emptyset$ such that every connected component of*

S contains at least one vertex from S_0 and for every vertex from S_0 all its neighbors on X which belong to S, are found in S_1. Such a partition we call a color of S (e.g., vertices from S_0 are colored into red and vertices from S_1 are colored into blue). We denote $\mu(S) = \max\{|S_0| : (S_0, S_1)$ is a color for $S\}$; a color (S_0, S_1) is called maximal if $|S_0| = \mu(S)$.

For maximal colors the component S_0 possesses some "minimax" property − it is a most dense set (for every $x \in S$ there is a neighbor belonging to S_0) among rare subsets of S (S_0 does not contain any two neighbor sites). In the following a numerical quantity $C_K(S)$, where $K \subseteq X$ is a given cluster and $S \subseteq X$ is arbitrary, is used. Since it is defined through the μ whose "minimax", subadditivity, and other properties (e.g., if one assumes that in the colors of S red vertices are positively charged and blue ones are neutral, $\mu(S)$ can be understood as some equilibrium distribution of the charge on S) remind us some notions from potential theory (e.g., [11]), we call C_K the (relative) capacity.

Definition 2 *Let $K, S \subseteq X$ be some clusters on X. The quantity*

$$C_K(S) = |K \setminus S| - (\mu(K) - \mu(K \cap S)) \tag{1}$$

is called the discrete capacity of S with respect to K.

Remark 1 *For arbitrary clusters $K, S \subseteq X$ the following is true: (1) $0 \leq C_K(S) \leq |K| - \mu(K)$; (2) if $K \subseteq S$ then $C_K(S) = 0$; (3) if $S \cap K = \emptyset$ then $C_K(S) = |K| - \mu(K)$; (4) if $S \subseteq K$ then both options $C_K(S) = 0$ and $C_K(S) > 0$ are possible (breakdown of monotony by inclusion – this differs C_K from such measures as set's volume or classical capacities).*

2.2. Networks and avalanches. The networks that we consider consist of weakly interacting (the intensity of interaction is small) phase-shifting rotators (PSR). A rotator consists of a particle P rotating on a circle C whose phase variable (in polar coordinate system with the origin at the center of C) is either decreasing or increasing function of time, depending on whether the rotation is clockwise or anti-clockwise (the angular velocity is negative or positive). A PSR is a perturbed uniform rotator in which the particle P rotates on C with constant angular velocity $\omega \neq 0$; it is required that the velocity remains unchanged under perturbations (a perturbation occurs when P receives a δ-kick) − this differs a PSR from the Chirikov-Taylor perturbed rotator considered in deterministic chaos. The interaction of PSR is as follows: when the rotating on C particle P hits its threshold ρ it sends a δ-kick to each of his neighbours; and a δ-kick received by a PSR causes an instantaneous rotation of the particle P on C on the angle α (without changing the direction of the rotation).

To give a formal definition of a PSR we identify it with its orbits; below, $\mathrm{N} = \{1, 2, \dots\}$ denotes the natural series, Z is the set of all integers, $\mathbf{i} = \sqrt{-1}$ is the imaginary unit, $\mathbf{C} = \{\frac{1}{2\pi}e^{\mathbf{i}\Phi} : 0 \le \Phi \le 2\pi\}$ is the circle of unit length centered at $z = 0$, and $h(t)$ is the Heaviside step-function: $h = 0$ if $t \le 0$ and $h = 1$ otherwise.

Definition 3 *Let α be a positive number, $\omega \neq 0$, $t_n > 0$ be such that $t_{n+1} - t_n > \eta$ for some $\eta > 0$ and all $n \in \mathrm{N}$, and $k_n \in \mathrm{N}$ be bounded. Let for $\omega > 0$ the $L(t)$ be defined as*

$$L(t) = \omega\, t + \alpha \sum_{n \in \mathbf{N}} k_n h(t - t_n) \tag{2}$$

(and if $\omega < 0$ the "$+$" here is replaced by "-") where $t > 0$ is time variable. The map

$$R: \;\; t \to (2\pi)^{-1} \exp(\mathbf{i}\, L(t)) \tag{3}$$

is called a phase-shifting rotator (PSR) on $\mathbf{C}$. The time series $R^{(in)} = \{t_n : n \ge 1\}$ and $R^{(out)} = \{\tau_n : R(\tau_n) = \rho\}$ are defined as the input and output of the rotator R.

The coefficient α in (2) is understood as intensity of a single δ-kick and k_n is the multiplicity of δ-kicks arriving simultaneously at the moment t_n. If $\alpha = 0$ then R in (3) is the complex exponential (in physics − harmonic oscillator). The $L(t)$ in (2) is proportional to the argument of a particle P, which started the motion at the moment $t = 0$ rotates on $\mathbf{C}$ with the constant angular velocity ω; at each moment t_n it is receiving some external δ-kicks each of which results in an instantaneous rotation of P on the angle α on $\mathbf{C}$. One can also consider PSR assigned on circles C with arbitrary radii r; then (2) gains the form

$$L(t) = \omega r t + \alpha r \sum_{n \in \mathbf{N}} k_n h(t - t_n). \tag{4}$$

Let us give the formal definition of RN and avalanches. On X a topology (so-called geospatial or network topology) − a map σ assigning the neighborhood on X, is defined. We consider the collections $\mathcal{R} = \{R_1, \dots, R_N\}$ where R_i is a PSR assigned on a circle C_i with radius $r_i > 0$, P_i is a particle rotating on C_i with angular velocity ω_i, and ρ_i is a threshold mark on C_i. Without loss of generality one can suppose that $X = E_N$ where $E_N = [1, 2, \dots, N]$ is a segment of N integers − indeed, renumbering (arbitrarily) a given X as $X = \{x_1, x_2, \dots, x_N\}$ one can assume that the rotator R_i associated with $x_i \in X$ to the $i \in E_N$ is assigned. A collection $\mathcal{R}$ constitutes a network if for every i the linking condition

$$R_i^{(in)} = \bigcup_{j \in \sigma(i)} R_j^{(out)} \tag{5}$$

where the multiplicity of every entry of $R_i^{(in)}$ coincides with the total number of its occurrences in different $R_j^{(out)}$, $j \in \sigma(i)$, holds. Our definition of avalanches involves a time parameter $\tau > 0$ (the term "avalanche" used in this paper is the same as the "τ-avalanche" in [1, 2, 3]).

Definition 4 *A collection $\mathcal{R} = \{R_1, \ldots, R_N\}$ of PSR assigned on E_N is called a rotator network (RN), if the R_i possess a common coupling constant α and for every i the Eq. (5) holds. We say that at i-th site of E_N at moment t a single event occurs, if P_i hits the threshold ρ_i on C_i during the time interval $[t, t+\tau)$. The collection of all the single events which occur at the same moment is called an avalanche. The collection of all the sites of E_N at which a single event constituting a given avalanche occurs, is called the support of the avalanche.*

3. Avalanche Memory. In this Section the avalanche memory in rotator networks is defined and studied. The clusters (with respect to a given topology σ) on $X = \{x_1, x_2, \ldots, x_N\}$ are assumed to be the only information-related units. A question how by means of avalanches a given cluster $K \subseteq X$ among others can be identified (restored), is considered. The answer to this question is given by Theorem 4. The identification task is resolved by including into a network of independent rotators some interaction whose intensity can be arbitrarily small.

We consider the RN $\mathcal{R} = \{R_1, \ldots, R_N\}$ where associated with x_i R_i is a PSR on a circle C_i with radius $r_i > 0$, P_i is a particle rotating on C_i with angular velocity ω_i, and $\rho_i \in C_i$ is a threshold mark. To make possible analytical results obtaining, the following is assumed. The linear speeds $v_1, \ldots, v_N$ ($v_i = \omega_i r_i$) of P_i are assumed to be Z-independent (for arbitrary $h_i \in$ Z the relation $\sum_{i=1}^{N} h_i v_i = 0$ implies that all the h_i are zero) and α is not equivalent to any of the ratios v_i/v_j (a number a is equivalent to b, that is $a \sim b$, if $a = \dfrac{h_1 + h_2 b}{h_3 + h_4 b}$ for some $h_1, \ldots, h_4 \in$ Z). We let $\beta = \dfrac{\max\{\omega_i : 1 \le i \le N\}}{\min\{\omega_i : 1 \le i \le N\}} - 1$, $\omega = N^{-1} \sum_{i=1}^{N} \omega_i$, $\theta = \omega\tau$, and require that $\alpha + \beta + \theta$ and α/θ are small enough (we do not specify the upper bounds).

Three types of RN are considered: interactive RN (IRN), where the particles on every site of X interact with the particles from all of its neighbors, non-interactive RN (NRN), where no interaction between particles is allowed, and partially interactive RN (PRN), where both kinds of particles exist. While in [1, 2, 3] the IRN were studied, in this paper the NRN and PRN are of most interest. An NRN is treated as a "clean memory" where no item is memorized. This clean memory is a dynamical object which reflects the type of the considered dynamics and serves as a "dynamical space" where some memory-related PRN (the $\mathcal{R}(K)$ in next Definition) associated with clusters on X, are embedded.

Definition 5 *For a network $\mathcal{R}$ on X the $I(\mathcal{R})$ denotes the collection of all interactive vertices of X (for instance, $\mathcal{R}$ is an IRN iff $I(\mathcal{R}) = X$ and it is an NRN iff $I(\mathcal{R}) = \emptyset$). For a cluster $K \subseteq X$ the network (a PRN) for which $I(\mathcal{R}) = K$ is denoted $\mathcal{R}(K)$.*

To define the avalanche memory $\mathcal{M}_{K,\mathcal{S}}$ of a cluster K (with respect to a given collection of clusters $\mathcal{S}$ on X), the following statement is considered. Let on X

an NRN be given, $\mathcal{S}$ be a collection of clusters on X, and $K \in \mathcal{S}$. Then how by assigning a weak interaction between rotators of the NRN a given cluster K among the items of $\mathcal{S}$ by using the avalanches can be identified? The clusters from $\mathcal{S}$ can be arbitrary or satisfying certain restrictions (e.g., on their size, topology, etc. − see also such a notice at the end of this Section). Hence, we deal with the following recognition task: for a given NRN to construct a PRN whose avalanches identify a given cluster among the entries of $\mathcal{S}$. For example, in the Hopfield theory [12] a cluster is identified (learned) as follows: the interaction (connections) in a network and their strengths are constructed in such a way that some energy, assigned to K, appears to be minimal against the energy of the other clusters. In our case, given K we also vary the network parameters (θ and α) but instead of energy use a modified Shannon entropy (the H^* in Eq. (8)). The solution to our task (Theorem 4) is formulated in terms of discrete capacity C_K defined in Section 2.

In neural networks terminology an avalanche can be viewed as "firing" of a cluster (which is the avalanche's support) and the topology of interaction as well as network numerical parameters determine the firing frequency. To involve the notion of entropy we are based on next theorem which states that as time progresses the avalanche process in a network assigns some non-trivial frequencies to the clusters on X (below, mes is the Lebesgue measure):

Theorem 1 *Let $\mathcal{R}$ be an RN on X and $S \subseteq X$ be a cluster. There exists the limit*

$$\pi_S = \lim_{T\to\infty} \frac{mes\ (e(S,T))}{T} \qquad (and \quad \lim_{\substack{\theta\to 0 \\ \alpha/\theta\to 0}} \pi_S = 0) \tag{6}$$

where $e(S,T)$ is the collection of the moments $1 \le t \le T$ at each of which an avalanche with the support S occurs (it follows from Definition 4 that $e(S,T)$ is a union of some numerical intervals).

In what follows a cluster $K \subset X$ is given and we deal with the networks $\mathcal{R}(K)$ (Definition 5). We emphasize the dependence of a network $\mathcal{R}$ on its parameters by denoting $\mathcal{R} = \mathcal{R}_{\alpha,\theta}$ − such $\mathcal{R}_{\alpha,\theta}$ having different values of α and θ are assigned on the same X with the same topology σ and rotators. To define the entropy of clusters, which is determined by the avalanche process, we consider the Shannon entropy function

$$h(p) = p\log_2 p + (1-p)\log_2(1-p), \quad 0 < p < 1$$

and define the entropy H of a cluster S with respect to network $\mathcal{R}_{\alpha,\theta}$ as

$$H(\mathcal{R}_{\alpha,\theta}; S) = h(\pi_S) \tag{7}$$

where π_S is the frequency (6) from Theorem 1. Instead of π_S we use the "relative probabilities" $\pi^*_S = \theta^{-|S|}\pi_S$ (this is the ratio of the firing probabilities of S in $\mathcal{R}_{\alpha,\theta}$

and in NRN $\mathcal{R}_{0,\theta}$; for small enough θ and α/θ we have $0 \leq \pi_S^* \leq 1$), and instead of H in Eq. (7) we deal with its modification H^*,

$$H^*(\mathcal{R}_{\alpha,\theta}; S) = h(\pi_S^*). \tag{8}$$

The next statement links explicitly the capacity C_K with the entropy H^*:

Theorem 2 *Let K and S be some clusters on X. Then the relation*

$$C_K(S) = \lim_{\substack{\theta\to 0\\ \alpha/\theta\to 0}} \frac{\ln \dfrac{H^*(\mathcal{R}_{\alpha,\theta}; K)}{H^*(\mathcal{R}_{\alpha,\theta}; S)}}{\ln \dfrac{\alpha}{\theta}} \tag{9}$$

where $\mathcal{R}_{\alpha,\theta} = \mathcal{R}_{\alpha,\theta}(K)$, holds.

Definition 6 *Let $\mathcal{S}$ be a collection of clusters on X, $K \in \mathcal{S}$ be given, and $\mathcal{R}_{\alpha,\theta} = \mathcal{R}_{\alpha,\theta}(K)$. The collection $\mathcal{M}_{K,\mathcal{S}} = \{E\}$ of such clusters $E \in \mathcal{S}$ for which the relations*

$$0 < c_1 \leq \frac{H^*(\mathcal{R}_{\alpha,\theta}; K)}{H^*(\mathcal{R}_{\alpha,\theta}; E)} \leq c_2 < \infty \tag{10}$$

for some constants c_1 and c_2 and small enough θ and α/θ hold, is called the avalanche memory (AM) of the cluster K (with respect to $\mathcal{S}$).

Particularly, K is found in its AM, $K \in \mathcal{M}_{K,\mathcal{S}}$, which is thus always non-empty. The next Theorem3 explains our definition of AM — it is required by Definition 6 that the items E of the memory $\mathcal{M}_{K,\mathcal{S}}$ should possess such a property: in the network $\mathcal{R}(K)$ by varying its numerical parameters θ and α the entropy H^* of E can be made for arbitrary times lesser than the entropy H^* of any other cluster S from $\mathcal{S}$:

Theorem 3 *Let a cluster $K \subseteq X$ be given and $\mathcal{R}_{\alpha,\theta} = \mathcal{R}_{\alpha,\theta}(K)$. For every cluster $S \subseteq X$ there exists the limit*

$$L = \lim_{\substack{\theta\to 0\\ \alpha/\theta\to 0}} \frac{H^*(\mathcal{R}_{\alpha,\theta}; K)}{H^*(\mathcal{R}_{\alpha,\theta}; S)} \tag{11}$$

and $L = 0$ if $C_K(S) > 0$ and $L > 0$ if $C_K(S) = 0$.

The next theorem (follows from Theorem 3) is our main statement — it gives the analytical description of AM of clusters K in terms of capacity C_K:

Theorem 4 *Let $\mathcal{S}$ be a collection of clusters on X and $K, E \in \mathcal{S}$. Then $E \in \mathcal{M}_{K,\mathcal{S}}$ if and only if*

$$C_K(E) = 0. \tag{12}$$

Regarding the $\mathcal{S}$ the following should be noticed. If $\mathcal{S}$ is the collection of all clusters on X and E is arbitrary, the Eq. (12) itself is not much restrictive − for instance, it holds for every $E \subseteq X$ containing K. The condition (12) is effective under some restrictions specifying the $\mathcal{S}$. For instance, let $\mathcal{S}$ be such that for arbitrary pair $S, S' \in \mathcal{S}$ we have $|S| = |S'|$; then (12) implies that if $K \subseteq E$ then $E = K$; it also follows that $|E| = |K|$ and then in the above-given formulations the H^* can be replaced by the Shannon entropy H.

The above-presented asserts that the avalanche process in RN is capable to operate as a kind of memory, dynamically restoring at future moments the clusters (patterns) memorized in the past. For instance, to restore a cluster $K \subseteq X$ in IRN, it suffices to reset the coupling constant α to 0 outside K, assign the positive parameters θ and α of the obtained PRN $\mathcal{R}(K)$ in such a way that θ and α/θ are small enough, and then (if following Definition 6) observe the avalanches in $\mathcal{R}(K)$ by computing their frequencies and entropies. In contrast, Theorem 4 gives us an analytical description of the AM,

$$\mathcal{M}_{K,\mathcal{S}} = \{E \in \mathcal{S} : C_K(E) = 0\} \tag{13}$$

which in computational sense concerns not the probabilities and entropies but the topology (IRN's topology σ restricted on K), colors, and capacities (e.g., if following Definition 2 or applying Remark 1). Note also, that since $C_K(K) = 0$ (Remark 1), the condition $C_K(E) = 0$ in Eq. (13) could be understood as reflecting the accuracy of the restoring of K by means of avalanches.

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Avalanches and Memory in Rotator Networks

A mechanism for operating of a memory in some dynamical networks (rotator networks, RN) is proposed. An RN is an abstract set X on which a neighbourhood relation is assigned and with every entry of X a phase-shifting rotator (PSR) is associated. A PSR is a perturbed uniform rotator (a particle P uniformly rotating on a circle C) where the perturbation is some instantaneous kicks which change the position of P on C but do not affect its angular velocity. An avalanche in RN is an event of almost simultaneous

attainment of some threshold marks on circles C by a number of particles P. The memory that we suggest operates on the basis of avalanches and is called the avalanche memory. A capacity for clusters on X is defined and in its terms the cluster identification task by means of avalanches happening in the network is considered.

А. Ю. Шахвердян

Лавины и память в ротаторных сетях

Предлагается механизм оперирования памяти в некоторых интерактивных динамических сетях, состоящих из абстрактного множества X на котором задано отношение соседства и с каждым элементом которого ассоциирован ротатор (частица P вращающаяся на некоторой окружности C) возмущаемый дельта-толчками. Такой толчок сдвигает частицу вдоль C на малый положительный угол, оставляя неизменной ее угловую скорость; взаимодействие между соседними ротаторами осуществляется посредством обмена дельта-толчками. Предлагаемая память определяется на основе лавин (лавина в сети есть почти одновременное достижение некоторой совокупностью частиц P заданных пороговых меток на соответсвующих окружностях C). Вводится емкость кластеров на X и в этих терминах решается задача идентификации (распознавания) кластеров посредством лавин.

Ա. Յու. Շահվերդյան

Հոսքեր եւ հիշողություն ռոտատորային ցանցերում

Առաջարկվում է հիշողության մեխանիզմ որոշ ինտերակտիվ դինամիկ ցանցերում: Դիտարկվող ցանցերը կազմված են X բազմությունից, որի վրա տրված է ցանցային տոպոլոգիա, եւ որի ամեն մի էլեմենտի հետ ասոցացված է դելտա-ռոտատոր - P մասնիկ, հավասարաչափ պտտվող C շրջանագծի վրա, որը ենթակա է թույլ արտաքին հարվածների, որոնք շեղում են P-ի դիրքը՝ C-ի վրա թողնելով անփոփոխ անկյունային արագությունը: Ցանցում գտնվող ամեն մի ռոտատոր փոխազդեցության մեջ է իր հարակիցների հետ: Առաջարկվող հիշողությունը պայմանավորված է ցանցում առկա հոսքերի գոյությամբ: Հոսքը մի պատահար է, երբ որոշակի քանակության P մասնիկներ համարյա միաժամանակ հատում են C-ի վրա նշված շեմային արժեքները: Սահմանվում է կլաստերների ունակություն, որի տերմիններով էլ ուսումնասիրվում է կլաստերների նույնականության (իդենտիֆիկացիայի) խնդիրը՝ հիմնվելով ցանցում հոսքերի առկայության վրա:

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