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Finite Difference Scheme for Two-Phase Obstacle Problem

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Key words: *free boundary, two-phase membrane, two-phase obstacle, finite difference*

Introduction. Mathematical setting of the problem and known results. Let $n \geq 2$ and $\Omega \subset \mathrm{R}^n$ be a bounded open subset with Lipschitz-regular boundary. Suppose we are given functions $g : \partial\Omega \to \mathrm{R}$, $\lambda^+, \lambda^- : \Omega \to \mathrm{R}$ such that g is continuous and takes both positive and negative values over $\partial\Omega$, and $\lambda^\pm$ are Lipschitz-continuous functions satisfying

$$\lambda^+(x) \geq 0, \quad \lambda^-(x) \geq 0, \quad \text{and} \quad \lambda^+(x) + \lambda^-(x) > 0, \quad x \in \Omega.$$

The *two-phase obstacle problem*, or the *two-phase membrane problem*, consist of minimizing the cost functional

$$\mathrm{J}(v) := \int_\Omega \left[\frac{1}{2}|\nabla v|^2 + \lambda^+ \max(v,0) + \lambda^- \max(-v,0)\right] dx \tag{1}$$

over the set of admissible "deformations" $\mathrm{K} := \{v \in H^1(\Omega) : \ v - g \in H_0^1(\Omega)\}$.

It is straightforward to see that the J is coercive, convex and lower-semicontinuous over $H^1(\Omega)$, resulting the existence of the unique minimum point u of the functional on the affine subspace $\mathrm{K} \subset H^1(\Omega)$.

Writing down the Euler-Lagrange equation for (1), we'll obtain

$$\begin{cases} \Delta u = \lambda^+ \cdot \chi_{\{u>0\}} - \lambda^- \cdot \chi_{\{u<0\}}, & x \in \Omega \\ u = g, & x \in \partial\Omega, \end{cases} \tag{2}$$

where χ_A stands for the characteristic function of the set A. It is easy to see (cf. [1]), that the solution (in the weak sense) of (2) must coincide with the minimizer $u \in \mathrm{K}$ of (1).

Problem (2) is an example of Free Boundary Problem. Roughly speaking, we have to solve $\Delta u = \lambda^+$ on the set $\{u > 0\}$ and $\Delta u = -\lambda^-$ on $\{u < 0\}$, but the sets $\{u > 0\}$ and $\{u < 0\}$, the two *phases* for this problem, are not known a priori, and need to be determined along the solution u. The free boundary for this problem consist of two parts- $\partial\{u > 0\} \cap \Omega$ and $\partial\{u < 0\} \cap \Omega$.

The *two-phase obstacle problem* (1) has been studied from different viewpoints. As it has been mentioned above, the existence of minimizers is straightforward and is obtained by the direct methods of calculus of variations. The optimal $C^{1,1}_{loc}$ regularity for the solution to (2) has been proved by Ural'tseva [2] in the case when the coefficients $\lambda^\pm$ are assumed to be constant, and the result was extended by Shahgholian [3] for Lipschitz-regular coefficients and by Lindgren, Shahgholian and Edquist [4] for Hölder-regular coefficients. The regularity and the geometry of the free boundary has been studied by Shahgholian, Ural'tseva and Weiss [5], [6], Andersson, Matevosyan and Mikayelyan [7].

As to numerical solution of two-phase obstacle problem, Bozorgnia in his recent paper [8] discussed three algorithms for numerical solution of two-phase obstacle problem. The first algorithm constructs an iterative sequence converging towards the solution. The second algorithm uses the regularization method to construct an approximation for the solution, and the third is based on Finite Element Method. But here the first and the third methods lack of convergence proofs, and for the second method only estimates for the difference between the regularized solutions and exact solution are given.

Here, in this paper, we use the regularization method to obtain a smooth approximation for two-phase obstacle problem, approximate the later by Finite Difference Scheme (FDS).

1. Finite difference scheme. Degenerate elliptic equations and viscosity solutions. Let Ω be an open subset of R^n, and for twice differentiable function $u : \Omega \to \mathrm{R}$ let Du and D^2u denote the gradient and Hessian matrix of u, respectively. Also let the function $F(x, r, p, X)$ be a continuous real-valued function defined on $\Omega \times \mathrm{R} \times \mathrm{R}^n \times S^n$, with S^n being the space of real symmetric $n \times n$ matrices. Write

$$\mathrm{F}[u](x) \equiv F\left(x, u(x), Du(x), D^2u(x)\right).$$

We consider the following second order fully nonlinear partial differential equation:

$$\mathrm{F}[u](x) = 0, \qquad x \in \Omega. \tag{3}$$

Definition 1. *The equation* (3) *is* ***degenerate elliptic*** *if*

$$F(x, r, p, X) \le F(x, s, p, Y) \quad \textit{whenever } r \le s \quad \textit{and} \quad Y \le X,$$

where $Y \le X$ means that $X - Y$ is a nonnegative definite symmetric matrix.

Definition 2. *u is called* ***viscosity subsolution*** *of* (3)*, if it is upper semicontinuous and for each $\varphi \in C^2(\Omega)$ and local maximum point $x_0 \in \Omega$ of $u - \varphi$ we have*

$$F\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \le 0.$$

Definition 3. *$u : \Omega \to \mathrm{R}$ is called* ***viscosity supersolution*** *of* (3)*, if it is lower semicontinuous and for each $\varphi \in C^2(\Omega)$ and local minimum point $x_0 \in \Omega$ of $u - \varphi$ we have*

$$F\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \ge 0.$$

Definition 4. *$u : \Omega \to \mathrm{R}$ is called* ***viscosity solution*** *of* (3)*, if it is both viscosity subsolution and supersolution (and hence continuous) for* (3).

The notion of viscosity solution was first introduced in 1981 by Crandall and Lions (see [9] and [10]) for first order Hamilton-Jacobi equations. It turns out that this notion is an effective tool also in the study of second order (elliptic and parabolic) fully nonlinear problems. There is a vast literature devoted to viscosity solutions by now, and for the general theory the reader is referred to [11], [12] and references therein.

Min-Max reformulation of the problem. Now we consider the following nonlinear problem, which we will refer as the *Min-Max form of the two-phase obstacle problem*:

$$\begin{cases} \min\left(-\Delta u + \lambda^+, \max(-\Delta u - \lambda^-, u)\right) = 0, & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4}$$

If we introduce a function $F : \Omega \times \mathrm{R} \times \mathrm{R}^n \times S^n \to \mathrm{R}$ by

$$F(x, r, p, X) = \min(-trace(X) + \lambda^+, \max(-trace(X) - \lambda^-, r)),$$

then the equation in (4) can be rewritten as

$$\mathrm{F}[u](x) = F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega, \tag{5}$$

and by **solution to** (4) we mean a function $u \in C(\overline{\Omega})$ which is a viscosity solution to (5) in the above-mentioned sense and satisfies $u = g$ along the boundary $\partial\Omega$.

It is easy to see that equation (5) is degenerate elliptic.

The following Propositions shows the connection between the problems (4) and (2). The first part of this proposition can be easily verified by using corresponding definitions, and the second part can be found in [1].

Proposition 1. *If u is the solution (in the weak sense) to* (2)*, then it is a viscosity solution to* (4)*. Moreover, u a.e. satisfies* (4)*.*

Uniqueness of the discrete solution. Now we are going to construct a Finite Difference Scheme (FDS) for one- and two-dimensional two-phase obstacle problems basing on the Min-Max form (4). For the sake of simplicity, we will assume that $\Omega = (-1, 1)$ in one-dimensional case and $\Omega = (-1, 1) \times (-1, 1)$ in two-dimensional case in the rest of the paper, keeping in mind that the method works also for more complicated domains.

Let $N \in \mathbb{N}$ be a positive integer, $h = 2/N$ and

$$x_i = -1 + ih,\ y_i = -1 + ih, \quad i = 0, 1, ..., N.$$

We are interested in computing approximate values of the two-phase obstacle problem solution at the grid points x_i or (x_i, y_j) in one- and two-dimensional cases, respectively. We will develop the one-dimensional and two-dimensional cases parallelly in this section, hoping that the same notations for this two cases will not make confusion for reader. We use the notation u_i and $u_{i,j}$ (or simply u_α, where α is one- or two-dimensional index) for finite-difference scheme approximation to $u(x_i)$ and $u(x_i, y_j)$, $\lambda_i^\pm = \lambda^\pm(x_i)$ and $\lambda_{i,j}^\pm = \lambda^\pm(x_i, y_j)$, $g_i = g(x_i)$ and $g_{i,j} = g(x_i, y_j)$ in one- and two-dimensional cases, respectively, assuming that the functions g and $\lambda^\pm$ are extended to be zero everywhere outside the boundary $\partial\Omega$ and outside Ω, respectively. In this section we will use also notations $u = (u_\alpha)$, $g = (g_\alpha)$ and $\lambda^\pm = (\lambda_\alpha^\pm)$ (not to be confused with functions u, g and $\lambda^\pm$). Also we will write $(a_\alpha) \le (b_\alpha)$ in I if $a_\alpha \le b_\alpha$ for all $\alpha \in \mathrm{I}$.

Denote

$$\mathrm{N} = \{i:\ 0 \le i \le N\} \quad \text{or} \quad \mathrm{N} = \{(i, j):\ 0 \le i, j \le N\},$$

$$\mathrm{N}^o = \{i:\ 1 \le i \le N - 1\} \quad \text{or} \quad \mathrm{N}^o = \{(i, j):\ 1 \le i, j \le N - 1\},$$

in one- and two- dimensional cases, respectively, and

$$\partial \mathrm{N} = \mathrm{N} \setminus \mathrm{N}^o.$$

In one-dimensional case we consider the following approximation for Δ operator: for any node $i \in \mathrm{N}^o$,

$$L_h u_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

and for two-dimensional case we introduce the following 5-point stencil approximation for Δ operator:

$$L_h u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2}$$

for any $(i, j) \in \mathrm{N}^o$.

Applying the finite difference method to (4), we obtain the following nonlinear system:

$$
\begin{cases} \min(-L_h u_\alpha + \lambda_\alpha^+ \,, \max(-L_h u_\alpha - \lambda_\alpha^- \,, u_\alpha)) = 0, & \alpha \in \mathrm{N}^o, \\ u_\alpha = g_\alpha, & \alpha \in \partial \mathrm{N}. \end{cases} \tag{6}
$$

It is not clear a priori, that this system have a solution, or, in the case of existence, this solution is unique. To this end, we consider the following functional:

$$
J_h(v) = -\frac{1}{2}\Big(L_h v, v\Big) + \Big(\lambda^+, v \vee 0\Big) - \Big(\lambda^-, v \wedge 0\Big) - \Big(L_h g, v\Big)
$$

defined on the finite dimensional space

$$
\mathrm{K} \;= \{v \in \mathrm{H} \;:\; v_\alpha = 0,\; \alpha \in \partial \mathrm{N}\},
$$

where

$$
\mathrm{H} \;= \{v = (v_\alpha) : v_\alpha \in \mathrm{R},\; \alpha \in \mathrm{N}\}.
$$

Here $v \vee 0 = \max(v, 0)$, $v \wedge 0 = \min(v, 0)$ and for $w = (w_\alpha)$ and $v = (v_\alpha)$, $\alpha \in$ N, the inner product $(\cdot, \cdot)$ is defined by

$$
(w, v) = \sum_{\alpha \in \mathrm{N}} w_\alpha \cdot v_\alpha.
$$

Lemma 1. *The element $u \in \mathrm{H}$ solves (6) if and only if $\tilde{u} = u - g$ solves the following minimization problem:*

$$
\tilde{u} \in \mathrm{K} \;: \qquad J_h(\tilde{u}) = \min_{v \in \mathrm{K}} J_h(v). \tag{7}
$$

Lemma 2. *The nonlinear system (6) has a unique solution.*

2. Convergence of approximation scheme. In this section we develop the convergence theory for the above-mentioned finite difference scheme for one- and two-dimensional cases. To do this, we first prove comparison principles for continuous and discrete equations, then, using the regularization technique, we obtain the error estimate for FDS.

Comparison principles for continuous and discrete nonlinear systems.

Lemma 3. *Let Ω be a bounded domain and $v_1, v_2 \in W^{2,\infty}(\Omega)$. If*

$$
\mathrm{F}[v_1] \le \mathrm{F}[v_2] \quad a.e.\ in \quad \Omega \qquad and \qquad v_1 \le v_2 \quad on \quad \partial\Omega,
$$

then

$$
v_1 \le v_2 \quad in \quad \Omega.
$$

Lemma 4. *Suppose $v_1, v_2 \in \mathrm{H}$. If*

$$\mathrm{F}_h[v_1] \le \mathrm{F}_h[v_2] \quad in \quad \Omega_h \qquad and \qquad v_1 \le v_2 \quad on \quad \partial\Omega_h,$$

then

$$v_1 \le v_2 \quad in \quad \Omega_h.$$

Regularization and error estimatation. The technique developed in this section applies for any dimension n. The idea comes from [13] and [14], where in the first article the author obtains some estimates for the rate of convergence of finite difference approximation for degenerate parabolic Bellman's equations, and in the second paper the method is developed to obtain the optimal convergence rate for finite difference approximation to American Option valuation problem.

Let $\beta \in C^\infty(\mathrm{R})$ be a function satisfying

$$\beta(z) = 1, \quad z \ge 1; \qquad \beta(z) = 0, \quad z \le -1;$$

$$\beta'(z) \ge 0, \quad z \in \mathrm{R}.$$

and $\beta_\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right)$, $x \in \mathrm{R}$. We denote by u^ε the solution of the following auxiliary problem:

$$\begin{cases} \Delta u^\varepsilon = \lambda^+ \cdot \beta_\varepsilon(u^\varepsilon) - \lambda^- \cdot \beta_\varepsilon(-u^\varepsilon) & in\ \Omega \\ u^\varepsilon = g & on\ \partial\Omega. \end{cases} \tag{8}$$

Lemma 5. *If u is the solution of two-phase membrane problem, and u^ε is the regularized solution (i.e. the solution of (8)), then*

$$|u - u^\varepsilon| \le \varepsilon.$$

Lemma 6. *If u^ε is the solution of (8), then*

$$|\mathrm{F}[u^\varepsilon]| \le \varepsilon \quad in \quad \Omega.$$

The next lemma plays an essential role in obtaining FDS approximation error estimate for regularized solution u^ε, since it is well-known that this error can be estimated using fourth-order partial derivatives of u^ε. For the proof of this lemma we need to impose a regularity assumption on $\lambda^\pm$.

Lemma 7. *If $\lambda^\pm \in C^3(\Omega)$, then there exists a constant $C > 0$, independent of ε, such that for every $i = 1, ..., n$,*

$$\left|\frac{\partial^4 u^\varepsilon}{\partial x_i^4}(x)\right| \le \frac{C}{\varepsilon^6}, \quad \forall x \in \Omega.$$

Lemma 8. *There exists a constant $C > 0$, independent of ε, such that*

$$|\Delta u^\varepsilon(x) - \Delta_h u^\varepsilon(x)| \le \frac{C}{\varepsilon^{9/2}} h^2, \quad \forall x \in \Omega.$$

To proceed, we denote by u_h the solution of

$$\begin{cases} \mathrm{F}_h[u_h] = 0, & \text{in } \Omega_h \\ u_h = g & \text{on } \partial\Omega_h. \end{cases} \tag{9}$$

From above mentioned lemmas we get the following theorem.

Theorem 1. *Let $\lambda^{\pm} \in C^3(\Omega)$, and u and u_h are the solutions of* (4) *and* (9)*, respectively. Then there exists a constant $K > 0$, independent of h, such that*

$$|u(x) - u_h(x)| \le K \cdot h^{2/7}, \quad x \in \Omega_h.$$

In particular, $u_h \to u$ as $h \to 0$.

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Метод конечных разностей для двухфазной задачи препятствий

Предлагается алгоритм решения *двухфазной задачи с препятствием* методом конечных разностей. Доказана сходимость метода и получена оценка погрешности для приближения методом конечных разностей.

Ա.Գ. Առաքելյան, Ռ. Հ. Բարխուդարյան, Մ.Պ. Պողոսյան

Վերջավոր փարբերությունների մեթոդը երկփուլ խոչընդոտի խնդրի համար

Առաջարկվում է *երկփուլ խոչընդոտի խնդրի* մոտավոր լուծման վերջավոր փարբերությունների մեթոդը: Բերված է առաջարկվող մեթոդի սխալանքի գնահատականը եվ ապացուցված է մեթոդի զուգամիտությունը:

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Finite Difference Scheme for Two-Phase Obstacle Problem

An algorithm to solve the *two-phase obstacle problem* by finite difference method is proposed. An error estimate for finite difference approximation is obtained and the convergence of proposed algorithm is proved.

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