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Love Waves in a Structure with an Inhomogeneous Layer

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Introduction. Earthquakes cause at least two types of surface waves, one characterised by elliptical in-plane particle displacements, and the other (more destructive type) by transverse shear displacements. Love [1] proposed a simple model of the Earth that assumed a thin crust rigidly bonded onto a semi-infinite substrate, the crust and substrate both being isotropic, linear elastic solids. In this structure a horizontal shear wave can propagate with an amplitude that decays rapidly with depth in the substrate. This localization of the amplitude makes surface waves a subject of great interest in many applications, because the energy spreads mainly in two dimensions, unlike bulk waves where the energy spreads in three dimensions. Over the years Love's result were generalised to include anisotropy, inhomogeneity, piezoelectric coupling [2-6].

Stoneley [7] studied the problem of the propagation of Love-type waves in a homogeneous finite layer sandwiched between two isotropic semi-infinite half-spaces. He showed that if the wave length is not too large or the thickness of the middle layer not too thin, such waves can propagate in this structure. The case of an inhomogeneous finite layer sandwiched between two semi-infinite half spaces where the rigidity and density of the middle layer vary exponentially across the layer has been considered in [8]. The objective of this paper is to study the inhomogeneous problem where the transverse variations in the material properties of the intermediate layer follow different relationships.

1. Statement of the problem. Consider an elastic inhomogeneous layer $0 < y < h$ (medium 0) sandwiched between two half spaces $-\infty < y < 0$ (medium 1) and $h < y < \infty$ (medium 2). We consider an antiplane problem and assume without loss of generality that waves propagate along the positive direction of the x-axis and the y-axis is directed positively into medium 2. The elastic displacements are directed along the Oz direction and are functions of x, y and t alone. It is assumed that the material properties change slowly with y and all material parameters have the same functional variation:

$$\mu(y) = \mu_0 f(y),\ \ \rho(y) = \rho_0 f(y). \tag{1}$$

The governing equations in this case are [1]

$$\mu_1 \nabla^2 w^{(1)} = \rho_1 \frac{\partial^2 w^{(1)}}{\partial t^2}, \tag{2}$$

$$\frac{\partial}{\partial y}(\mu(y)\frac{\partial w^{(0)}}{\partial y}) + \mu(y)\frac{\partial^2 w^{(0)}}{\partial x^2} = \rho(y)\frac{\partial^2 w^{(0)}}{\partial t^2}, \tag{3}$$

$$\mu_2 \nabla^2 w^{(2)} = \rho_2 \frac{\partial^2 w^{(2)}}{\partial t^2}, \tag{4}$$

where $w^{(1)}, w^{(0)}, w^{(2)}$ are the z-displacements in media 1, 0 and 2 respectively, $\mu_1, \mu(y), \mu_2$ are the shear moduli, $\rho_1,\ \rho(y),\ \rho_2$ the mass densities, and $\nabla^2 = \partial/\partial x^2 + \partial/\partial y^2$. We introduce a new function $w_0(x, y, t)$ such that

$$w^{(0)} = \frac{w_0(x, y, t)}{\sqrt{f(y)}}. \tag{5}$$

From (5), equation (3) takes the following form with respect to the function w_0

$$\mu_0 \Delta w_0 - \mu_0 \frac{\sqrt{f(y)}''}{\sqrt{f(y)}} w_0 = \rho_0 \frac{\partial^2 w_0}{\partial t^2}. \tag{6}$$

Here we suppose that the inhomogeneity is such that the function $f(y)$ satisfies the following condition

$$\frac{\partial_{y,y}(\sqrt{f(y)})}{\sqrt{f(y)}} = a^2 = \mathrm{const}\,. \tag{7}$$

2. Inhomogeneous layer when a in (7) is real. When a is real, from (7) the function $f(y)$ can be written

$$f(y) = [A\cosh(\frac{\beta y}{h}) + B\sinh(\frac{\beta y}{h})]^2, \tag{8}$$

where $\beta = ah$ is a dimensionless parameter. We consider a particular case of (8) when the function $f(y)$ is taken as

$$f(y) = \cosh^2(\frac{\beta y}{h}). \tag{9}$$

In this case equation (6) for the inhomogeneous layer takes the following form

$$\Delta w_0 - \frac{\beta^2}{h^2} w_0 = \frac{\rho_0}{\mu_0} \frac{\partial^2 w_0}{\partial t^2}. \tag{10}$$

The solutions for (2), (4) and (10) attenuating when $y \to \pm\infty$ can be presented in the following way:

$$w^{(1)}(x, y, t) = C_1 e^{-kqy} e^{ik(x-ct)}, \tag{11}$$

$$w_0(x, y, t) = [C_2 \cos(kpy) + C_3 \sin(kpy)] e^{ik(x-ct)}, \tag{12}$$

$$w^{(2)}(x, y, t) = C_4 e^{kry} e^{ik(x-ct)}, \tag{13}$$

where

$$q = \sqrt{1 - \frac{c^2}{c_1^2}},\ r = \sqrt{1 - \frac{c^2}{c_2^2}},\ p = \sqrt{\frac{c^2}{c_0^2} - 1 - \frac{\beta^2}{\xi^2}}, \tag{14}$$

k is the wave number, $c = \omega/k$ is the surface wave phase velocity, $c_i^2 = \mu_i/\rho_i$ $(i = 1, 0, 2)$ are the shear wave velocities in the corresponding media, and $\xi^2 = k^2 h^2$.

Boundary conditions for the continuity of the stresses and displacements are:

1. at $y = h$

$$w^{(1)} = w_0,\ \mu_1 \frac{dw^{(1)}}{dy} = \mu_0 f(y) \frac{dw_0}{dy}, \tag{15}$$

2. at $y = 0$

$$w^{(2)} = w_0,\ \mu_2 \frac{dw^{(2)}}{dy} = \mu_0 f(y) \frac{dw_0}{dy}. \tag{16}$$

2.1 Propagation of surface waves with a sinusoidally varying amplitude in the inhomogeneous layer. It follows from (11) and (13) that a surface wave with an amplitude that varies sinusoidally across the inhomogeneous layer and decays to zero as $y \to \pm\infty$ propagates if $p^2 > 0$ and the following conditions take place:

$$c_0 \sqrt{\frac{\beta^2}{\xi^2} + 1} < c < \min\{c_1, c_2\}. \tag{17}$$

Substitution of solutions (11)-(13) into the boundary conditions (15) and (16) leads to the system of equations for determining the unknown constants. The solvability condition gives the following dispersion equations for defining the phase velocity of the surface wave:

$$\tan(\xi p) = \frac{\mu_{20} \xi pr Cosh(\beta) + \mu_{10} \xi pq Sech(\beta) - p\beta Sinh(\beta)}{\xi p^2 Cosh(\beta) - \mu_{10}\mu_{20} \xi qr Sech(\beta) + \mu_{20} r\beta Sinh(\beta)}, \tag{18}$$

where $\mu_{10} = \mu_1/\mu_0$ and $\mu_{20} = \mu_2/\mu_0$. To investigate the existence of roots of the dispersion equation (18) we rewrite it in the following form

$$F(c) = \frac{tan(\xi p)}{p} - \frac{\mu_{20} r + d}{p^2 - \mu_{20} r d} = 0,\ d = \mu_{10} q sech^2(\beta) - \frac{\beta}{\xi} tanh(\beta). \tag{19}$$

First assume that $c_2 < c_1$. Then when $c = c_m$ where $c_m = c_0\sqrt{(\beta/\xi)^2 + 1}$ then $p = 0$, $F(c_m) = -\xi\mu_{20} r d - \mu_{20} r - d$ and for a sufficiently small value of β the expression for d in (19) can be made positive. Since q and r are positive $F(c_m) < 0$. When $c = c_2$ $(r = 0)$ the value of $F(c)$ becomes positive if $tan(\xi p) > d/p$. Hence for sufficiently large ξ there is a root of equation (18) between c_m and c_2.

If $0 < \xi p < \pi/2$ then for the whole interval (17) $tan(\xi p)$ is increasing. Since $\dfrac{p(\mu_{20} r + d)}{p^2 - \mu_{20} r d}$ is decreasing and at $c = c_2$ has the value d/p, if $tan(\xi p) < d/p$ the dispersion equation (18) has no roots.

If $c_0\sqrt{(\beta/\xi)^2 + 1} < c < c_1 < c_2$ then $F(c)$ is negative at $c = c_m$. At $c = c_1$ the value of $F(c)$ is positive if $tan(\xi p) > g$, where

$$g = \frac{p(\mu_{20} r - (\beta/\xi) tanh(\beta))}{(p^2 + \mu_{20} r (\beta/\xi) tanh(\beta))}. \tag{20}$$

Hence for sufficiently large ξ the dispersion equation (19) has a root between c_m and c_1. If within the interval $c_0\sqrt{(\beta/\xi)^2 + 2} < c < c_1$, ξ is such that $\xi p < \pi/2$, and at $c = c_1$, $tan(\xi p) < g$, then there is no root and Love-type waves do not propagate.

2.2. Propagation of surface waves with a non-sinusoidally varying amplitude in the inhomogeneous layer. If a surface wave propagates when $p^2 = -p_0^2$, where $p_0 = \sqrt{1 + (\beta/\xi)^2 - c^2/c_0^2}$, its amplitude will vary non-sinusoidally in the middle layer and will have a velocity $c < c_0\sqrt{(\beta/\xi)^2 + 1}$ and $c < \min\{c_1, c_2\}$. The dispersion equation in this case takes the following form

$$tanh(\xi p_0) = -\frac{\mu_{20}\xi p_0 r Cosh(\beta) + \mu_{10}\xi p_0 q Sech(\beta) - p_0 \beta Sinh(\beta)}{\xi p_0^2 Cosh(\beta) + \mu_{10}\mu_{20}\xi q r Sech(\beta) - \mu_{20} r \beta Sinh(\beta)}. \tag{21}$$

It can be seen from (21) that when $\beta = 0$ the surface wave does not propagate in the structure with a homogeneous layer. However the detailed numerical analysis shows that when $\beta \neq 0$ the dispersion equation may have a single solution if

$$c_0 < \min\{c_1, c_2\} \text{and } \left(\frac{\beta}{\xi}\right)^2 > \left(\frac{\min(c_1, c_2)}{c_0}\right)^2 - 1. \tag{22}$$

If $c_0 < \min\{c_1, c_2\}$ but also $\left(\dfrac{\beta}{\xi}\right)^2 \leq \left(\dfrac{\min(c_1, c_2)}{c_0}\right)^2 - 1$ the dispersion equation (21) has no roots. If $c_0 \geq \min\{c_1, c_2\}$ the dispersion equation has no solutions for all values of β and ξ.

The numerical results show (Table 1, Table 2) that for all given values of β and ξ a surface wave with either sinusoidally or non-sinusoidally varying amplitude in the inhomogeneous layer exists.

3. Inhomogeneous layer when a in (7) is imaginary. Assuming now that the parameter a in (7) is imaginary, writing $a^2 = -\beta^2/h^2$ and supposing that $\beta < \dfrac{\pi}{2}$ and $f(y) = \cos^2\left(\dfrac{\beta y}{h}\right)$, for a surface wave with a sinusoidally varying amplitude in the inhomogenous layer and attenuating for $y \to \pm\infty$ the following conditions must take place:

$$c_0\sqrt{1-\frac{\beta^2}{\xi^2}} < c < \min\{c_1, c_2\}. \tag{23}$$

The dispersion equation takes the following form:

$$tan(\xi p) = \frac{G_{20}\xi prCos(\beta) + G_{10}\xi pqSec(\beta) + p\beta Sin(\beta)}{\xi p^2 Cos(\beta) - G_{10}G_{20}\xi qrSec(\beta) - G_{20}r\beta Sin(\beta) - G_{20}r\beta Sin(\beta)}, \tag{24}$$

or

$$F(c) = \frac{tan(\xi p)}{p}(p^2 - \mu_{20}rs) - \mu_{20}r - s = 0,\ \ s = \mu_{10}q sec^2(\beta) + \frac{\beta}{\xi}tan(beta). \tag{25}$$

If $c_2 > c_1$ and $c = c_m = c_0\sqrt{1-(\beta/\xi)^2}$ such that $p = 0$, then for all values of β for which $tan(\beta) > 0$, $s > 0$ and since q and r are positive then $F(c_m) < 0$. At $c = c_2$, $F(c) > 0$ if $tan(\xi p) > s/p$. This means there is a root for dispersion equation (26) between c_m and c_2 if ξ is made sufficiently large.

If is so small that for the whole interval $c_0 < c < c_2$, $\xi p < \pi/2$, and at $c = c_2$, $tan(\xi p) < s/p$ then the dispersion equation (26) has no roots.

If $c_0\sqrt{1-\beta^2/\xi^2} < c < c_1 < c_2$ then $F(c)$ is negative when $p = 0$ at $c = c_m$, and when $c = c_1$, $F(c)$ is positive if $tan(\xi p) > b$, where

$$b = \frac{p(\mu_{20}r + (\beta/\xi)tan(\beta))}{(p^2 - \mu_{20}r(\beta/\xi)tan(\beta))}.$$

Therefore, there are values of ξ for which the dispersion equation (26) has a root between c_m and c_1. If within the interval $c_0 < c < c_1$, ξ is so small that $\xi p < \pi/2$ and at $c = c_1$, $tan(\xi p) < b$, then there will be no root. Consequently, no Love-type wave will exist.

If $p^2 = -p_0^2$, where $p_0 = \sqrt{1 + (\beta/\xi)^2 - c^2/c_0^2}$, the following dispersion equation is obtained for the surface wave with a non-sinusoidally varying amplitude in the middle layer:

$$tan(\xi p) = -\frac{\mu_{20}\xi p_0 rCos(\beta)^2 + p_0\beta Sin(\beta)Cos(\beta) + \mu_{10}\xi p_0 q}{\xi p_0^2 Cos(\beta)^2 + \mu_{20}r\beta Sin(\beta)Cos(\beta) + \mu_{10}\mu_{20}\xi qr}. \tag{26}$$

Since the left hand side of the this equation is always positive and the right hand side always negative such waves do not exist.

4. Numerical results. Roots of the dispersion equation (19) have been calculated using the following values of the elastic constants and densities $(c_i(10^3 m/s), \rho_i(10^3 kg/m^3), \mu_i(10^{10} N/m^2))$.

1. $f(y) = cosh^2(\beta y/h)$

a) $c_2 < c_1$: $c_1 = 3.32,\ \rho_1 = 7.67,\ \mu_1 = 8.45$, $c_2 = 1.48, \rho_2 = 19.3,\ \mu_2 = 4.24$, $c_0 = 1.2, \rho_0 = 1.19, \mu_0 = 0.17$.

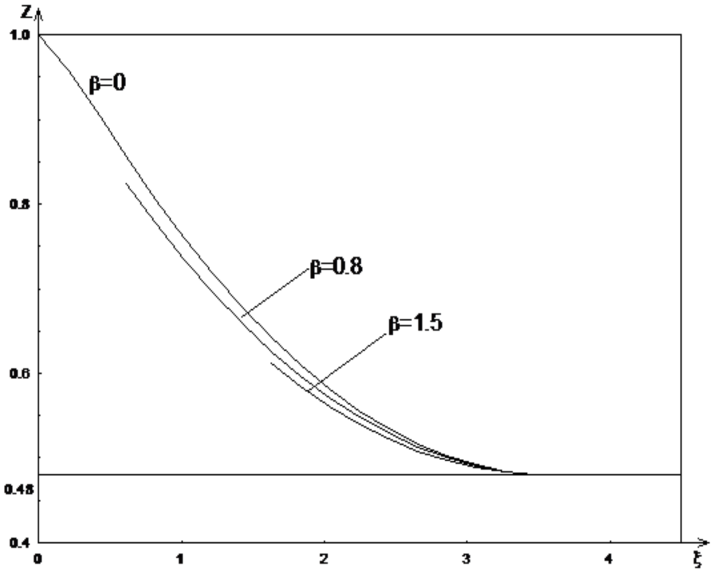

Fig. 1. Surface wave velocities for different values of the parameter β, for $f(y) = cosh^2(\beta y/h)$, $z = c/c_2$.

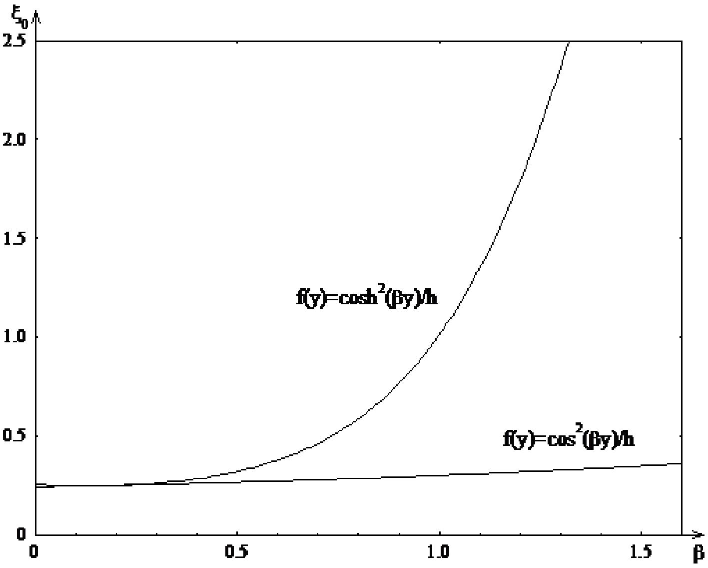

Fig.2.The dependence of β on the minimum value of the relative thickness of the middle layer below which surface waves do not occur.

Taking $\beta = 0.8$ and $\xi = 2.4$ gives $p = 0$, $c = c_m = 1.26$ and $F(c_m) = -820.7$. At $c = c_2$, $F(c) = 0.66$. This means that the dispersion equation (19) has a root between

$c = c_m$ and $c = c_2$.

b) $c_2 > c_1$: $c_1 = 1.48$, $\rho_1 = 19.3$, $\mu_1 = 4.24$, $c_2 = 3.32$, $\rho_2 = 7.67$, $\mu_2 = 8.45$, $c_0 = 1.2$, $\rho_0 = 1.19$, $\mu_0 = 0.17$.

If $\beta = 0.8$ and $\xi = 2.4$, $c = c_m = 1.26$ and $F(c) = -817$. At $c = c_1 = 1.48$, $F(c) = 573.4$. Hence there is a root of the dispersion equation between $c = c_m$ and $c = c_1$.

The dependence of the phase velocity on the relative thickness of the middle layer Table 1 (sinusoidal case) and Table 2 (non-sinusoidal case)

Table 1

ξ / β	0.25	0.3	0.5	0.6	0.8	1	1.5	1.8	2	∞
0	0.99	0.98	0.89	0.85	0.78	0.71	0.67	0.64	0.62	0.48
0.5	-	0.97	0.89	0.84	0.78	0.74	0.66	0.63	0.62	0.48
0.8	-	-	-	0.81	0.75	0.71	0.64	0.62	0.60	0.48
1.05	-	-	-	-	-	-	0.63	0.60	0.59	0.48
1.3	-	-	-	-	-	-	-	-		0.48

Table 2

ξ / β	0.25	0.3	0.5	0.6	0.8	1	1.5	1.8	2	∞
0	-	-	-	-	-	-	-	-	-	-
0.5	0.99	-	-	-	-	-	-	-	-	-
0.8	0.98	0.96	0.85	-	-	-	-	-	-	-
1.05	0.96	0.93	0.82	0.79	0.73	0.69	-	-	-	-
1.3	0.93	0.89	0.78	0.75	0.70	0.65	0.64	0.59	0.58	-

The results of numerical calculations of the Love-type wave velocities of the first mode normalized against the shear wave velocity of the upper half-space $(c_2 < c_1)$ are shown in Figure 1. As β increases the surface wave starts propagating with smaller velocity and for bigger values of the relative thickness. The velocities decrease for shorter waves approaching to the limiting value c_0. Figure 2 shows the relationship between the inhomogeneity parameter and the minimum value of the relative thickness below which surface waves with sinusoidaly varying amplitude in the middle layer do not occur. The influence of the inhomegeneity is much stronger when $f(y) = cosh^2(\beta y/h)$ than for the case $f(y) = cos^2(\beta y/h)$.

The dependence of the phase velocity of the Love-type wave on the relative thickness of the middle layer for surface waves are shown in Table 1 (sinusoidal case) and Table 2 (non-sinusoidal case). According to the tables a surface wave with a non-sinusoidally varying amplitude cannot propagate in a homogeneous layer. With increasing inhomogeneity, surface waves with a sinusoidally varying amplitude occur for larger values of the relative thickness of the layer. For the same value of the inhomogeneity parameter a surface wave with non-sinusoidally varying amplitude always occurs for thinner middle layers.

2. $f(y) = cos^2(\beta y/h)$

Here $c_2 < c_1$ and $c_1 = 3.32$, $\rho_1 = 7.67$, $\mu_1 = 8.45$, $c_2 = 1.48$, $\rho_2 = 19.3$, $\mu_2 = 4.24$, $c_0 = 1.2$, $\rho_0 = 1.19$, $\mu_0 = 1.71$.

For $\beta = 0.8$, $\xi = 1.85$, $p = 0$ when $c = c_m = 1.08$ and $F(c_m) = -3133.54$. At $c = c_2 = 1.48$, $F(c) = 1.48$. Hence there is a root of the dispersion equation between $c = 1.08$ and $c = 1.48$.

Table 3

Love-type wave velocities for for different values of the inhomogeneity parameter β ($c_2 < c_1$ and the figures are normalized against c_2)

ξ / β	0.25	0.3	0.5	0.6	0.8	1	1.5	1.8	2	2.4	2.8	∞
0	0.99	0.97	0.89	0.85	0.79	0.71	0.66	0.64	0.62	0.6	0.58	0.48
0.5	-	0.99	0.91	0.87	0.8	0.75	0.67	0.64	0.62	0.6	0.58	0.48
1.05	-	1	0.93	0.89	0.82	0.77	0.68	0.65	0.63	0.60	0.58	0.48
1.5	-	0.97	0.88	0.84	0.77	0.73	0.65	0.62	0.61	0.59	0.57	0.48

Table 3 shows that the first mode of the Love-type wave for $f(y) = cos^2(\beta y/h)$ almost always starts propagating when the relative thickness has the value $\xi \approx 0.3$ ($\beta < \pi/2$).

5. Conclusion. The problem of propagation of Love-type waves in an inhomogeneous finite layer sandwiched between two isotropic half spaces is investigated. The problem is considered for two types of inhomogeneity, $cosh^2(\beta y/h)$ and $cos^2(\beta y/h)$, assuming that the material properties change gradually with the thickness of the layer and all material parameters have the same functional variation.

It is shown that for the case $cosh^2(\beta y/h)$ a Love-type wave with a sinusoidally varying amplitude in the middle layer can propagate if the relative thickness of the middle layer is sufficiently large or the wave length is sufficiently short. The results also show that for this type of inhomogeneity, unlike the homogeneous case, a surface wave with a non-sinusoidally varying amplitude in the middle layer can also propagate. These waves occur when the middle layer is relatively thin or for longer wavelengths. Thus if for a homogeneous case a surface wave exists only for a sufficiently thick middle layer [7] here a surface wave exists for all values of the relative thickness of the middle layer.

In the case $f(y) = cos^2(\beta y/h)$ the surface wave starts propagating almost always at the same value of the relative thickness $\xi \approx 0.3$. Thus the dependence on the inhomogeneity is weaker here. This is due to the assumption for the inhomogeneity parameter $\beta < \pi/2$ which does not allow the mechanical properties of the layer to become zero.

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Love Waves in a Structure with an Inhomogeneous Layer

The problem of the propagation of Love type waves in a structure consisting of a finite inhomogeneous layer sandwiched between two isotropic homogeneous half spaces is investigated. Two types of inhomogeneity are considered. It is shown that in one case the amplitude of vibrations in the middle layer is a sinusoidal function of distance from the plane of symmetry, but that in the other case it may be non-sinusoidal for certain values of the parameters of the problem.

Կ. Բ. Ղազարյան, Դ. Գ. Փիլիպոսյան

Լյավի փիպի ալիքներ անհամասեռ շերտով կառուցվածքում

Անհամասեռ շերտից եւ կից երկու առաձգական կիսապարածություններից կազմված կառուցվածքում դիտարկված է Լյավի փիպի ալիքների պարածումը: Դիտարկված է երկու փիպի անհամասեռություն: Ցույց է պրված, որ մի դեպքում հնարավոր է նաեւ Լյավի ընդհանրացված փիպի ալիքների պարածումը շերտում ոչ սինուսոիդալ փոփոխվող ամպլիտուդով:

К. Б. Казарян, Д. Г. Пилипосян

Волны Лява в структуре с неоднородным слоем

Рассмотрена задача распространения волны типа Лява в слоистой системе, состоящей из неоднородного слоя, находящегося в условиях контакта с двумя однородными подпространствами. Рассмотрено два типа неоднородностей.

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