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Miscellanea in Integral Geometry, Algebraic Geometry and Real Analysis

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Three identities for the integrals along curves. We give first three mutually connected identities for the integrals $\int_\Gamma K dl$ for "good" curves Γ and function K on Γ. One of them generalizes the key identity in integral geometry, the Crofton's formula, which we obtain taking $K \equiv 1$. The identities are very simple but we have not seen them anywhere.

To obtain these results we make use, in fact, some reasonings we applied in 1970s to study Gamma-lines of meromorphic functions, see [1], also [2] and [3]. Much later we recognized that a part of these reasonings is almost identical with the key method in integral geometry. Then our task was simply to present these reasonings in terms of integral geometry, what we do below.

1.1. Two identities related to the integrals along curves. The notations of the previous sections have been used but for the reader interested only in this chapter we repeat the notations.

Let $\bar{X}(\theta)$ be the oriented straight line passing through zero and having direction θ, that is $\bar{X}(\theta) := \{(x,y) | \ \theta := \arctan(y/x)\}$. We will use notation $X(\theta)$ for the coordinate on $\bar{X}(\theta)$. Denote by $J_{X(\theta)}$ the straight line composing the angle $\theta + \pi/2$ with x-axis and passing through the point on $\bar{X}(\theta)$ with the coordinate $X(\theta)$.

In what follows curve means an oriented plane curve. Let Γ be a curve with continuous tangent and finite length. Thus we can define the acute angle δ that composes the tangent to Γ with x-axis. Denote by $N(\Gamma \cap J_{X(\theta)})$ the number of elements of the set $\Gamma \cap J_{X(\theta)}$. Observe that either these element are points (denote them by $(X(\theta), Y_i(\theta))$ or these elements are common parts of Γ and $J_{X(\theta)}$, which should be then some intervals on $\Gamma \cap J_{X(\theta)}$). We prescribe to the points $(X(\theta), Y_i(\theta)$ the weight $\omega_i(X(\theta)) = K(X(\theta), Y_i(\theta)) \cos(\delta(X(\theta), Y_i(\theta)) - \theta)$ and prescribe to the intervals (on which $\cos(\delta(X(\theta), Y_i(\theta)) - \theta) = 0$) the weight $\omega_i(X(\theta)) = 0$. Thus we are able to define the following weight function $W_{\Gamma,K}(X(\theta))$ of variable $X(\theta)$:

$$W_{\Gamma,K}(X(\theta)) := \left\{ \sum_{i=1}^{N(\Gamma \cap J_{X(\theta)})} \omega_i(X(\theta)) \right\}.$$

Denote by $\Gamma \perp \bar{X}(\theta)$ the (ordinary) orthogonal projection of Γ on $\bar{X}(\theta)$

All the integrals that are used in this chapter are in the sense of Lebesgue.

Identity 1.1. *For any Γ with continuous tangent and finite length and any continuos function K on Γ we have*

$$\int_{\Gamma} K dl = \frac{2}{\pi} \int_0^{\pi} \int_{\Gamma \perp \bar{X}(\theta)} W_{\Gamma,K}(X(\theta)) dX(\theta) d\theta.$$

We obtain also a similar identity which make use of the projections on the straight lines $\bar{X} := X(0)$ and $\bar{Y} := \bar{X}(\pi/2)$ (instead of integration in θ).

Identity 1.2. *For any Γ with continuous tangent and finite length and any continuos function K on Γ we have*

$$\int_{\Gamma} K dl = \int_{\Gamma \perp \bar{X}(0)} W_{J_{X(0)}}(\Gamma, K) dX(0) + \int_{\Gamma \perp \bar{X}(\pi/2)} W_{J_{X(\pi/2)}}(\Gamma, K) dX(\pi/2).$$

1.2. An identity generalizing the key identity in integral geometry. Due to the key Crofton's identity in integral geometry we have for the above defined curve:

$$l(\Gamma) = \frac{1}{2} \int_0^{\pi} \int_{\Gamma \perp \bar{X}(\theta)} N(\Gamma \cap J_{X(\theta)}) dX(\theta) d\theta,$$

see [12], formula (3.17).

Now we prescribe to the elements of the set $\Gamma \cap J_{X(\theta)}$ the weight $\omega_i^*(X(\theta)) = K(X(\theta), Y_i(\theta))|$ when these elements are points and prescribe the weight $\omega_i^*(X(\theta)) = 0$ if these elements are common parts of Γ and $J_{X(\theta)}$. We define a new weight function $W^*_{\Gamma,K}(X(\theta))$ of the variable $X(\theta)$ defined on $\Gamma \perp \bar{X}(\theta)$:

$$W^*_{\Gamma,K}(X(\theta)) := \left\{ \sum_{i=1}^{N(\Gamma \cap J_{X(\theta)})} \omega_i^*(X(\theta)) \right\}.$$

Identity 1.3. *For any Γ with continuous tangent and finite length and any continuos function K on Γ we have*

$$\int_{\Gamma} K dl = \frac{1}{2} \int_0^{\pi} \int_{\Gamma \perp \bar{X}(\theta)} W^*_{\Gamma,K}(X(\theta)) dX(\theta) d\theta.$$

Notice that for $K \equiv 1$ Identity 1.3 implies the Crofton's identity.

2. Some identities and inequalities connecting the length and curvature of curves. As above we assume that $\Gamma \in C^2$ is an oriented curve so that we can define the tangential angle β, respectively curvature $k(l)$ at any point $l \in \Gamma$, and can define the magnitude $C(\Gamma) := \int_\Gamma |k(l)|\, dl$ called usually *absolute integral curvature* of Γ. The magnitude $C(\Gamma)$ plays a crucial role in many pure and applied studies.

The problems we consider are closely connected with the known Fáry's inequality (see [6] also book [12], formula (3.26)) which asserts that *for any closed curve* $\Gamma^* \in C^2$ we have $l(\Gamma^*) \leq \frac{1}{2}\mathrm{diam}\Gamma^* C(\Gamma^*)$. This inequality bore a lot of generalizations for very different objects in geometry (see Internet). The inequality is sharp, but only when Γ^* are circumferences.

In this section we consider the following problems. Are there similar inequalities for the closed curves that are sharp for rather large classes of curves? What can be said about non closed case? And much more interesting seemingly problem: are there some identities where the length and the curvature of a given curve occur simultaneously?

2.1. An improvement and a complement of the Fary's inequality. Clearly for a given curve we cannot have an identity determined merely by the length and curvature: some additional notions are needed for that. We utilize the following notion of *rotational length* of Γ, that is $\Delta(\Gamma) := \frac{1}{2}\int_0^\pi l(\Gamma \perp \bar{X}(\theta))d\theta$.

Inequality 2.1. *For any closed curve $\Gamma^* \in C^2$ we have*

$$l(\Gamma^*) \leq \frac{1}{2}\mathrm{diam}\Gamma^* C(\Gamma^*) + 2\Delta(\Gamma^*) - \pi\mathrm{diam}\Gamma^*.$$

This inequality implies the Fary's inequality since $2\Delta(\Gamma^*) - \pi\mathrm{diam}\Gamma^* \leq 0$. Sharpness. Due to the Identity 2.1 below Inequality 2.1 is sharp for any closed convex curve. This shows that Inequality 2.1 improves Fary's inequality essentially.

Inequality 2.2. *For any non closed curve $\Gamma \in C^2$ we have*

$$l(\Gamma) \leq \frac{1}{2}\mathrm{diam}\Gamma C(\Gamma) + \Delta(\Gamma).$$

Sharpness. This is rather rough inequality which meantime is sharp for any segment. During the proofs we establish more sharp inequality.

2.1. Identities for closed and non closed curves. Saying convex curve Γ we mean that the intersection of Γ with any straight line consists of at most two elements which can be as points as well as segments.

An important observation. It is enough to obtain an identity where occurs the absolute integral curvature for convex curves. Indeed, since Γ (closed or non closed) can be represented as a union $\cup_i \Gamma_i$ of convex (non closed) sub curves Γ_i we have $C(\Gamma) = \sum_i C(\Gamma_i)$ so that we can apply similar identity to Γ_i and then sum up.

Identity 2.1. *For any convex closed curve* $\Gamma^* \in C^2$ *we have*

$$l(\Gamma^*) = \frac{1}{2}\mathrm{diam}\Gamma^* \left[C(\Gamma^*) - 2\pi\right] + 2\Delta(\Gamma^*).$$

Let a and b be the endpoints of a given convex non closed curve $\tilde\Gamma \in C^2$ *and* I be a segment connected a and b. We can consider the union $\tilde\Gamma \cup I$ as a curve and notice that $\tilde\Gamma \cup I$ can be considered as an oriented closed convex curve γ. The curve γ has, in general case, two jumps of the tangential angles at the points a and b; we denote these jumps respectively by α and β. (This is true also when $\tilde\Gamma$ is a segment. Then γ is the curve which passes this segment twice in opposite directions; respectively $\alpha = \beta = \pi$).

Identity 2.2. *For any convex non closed curve* $\Gamma \in C^2$ *we have*

$$l(\Gamma) = \frac{1}{2}\mathrm{diam}\Gamma \left[C(\Gamma) + \alpha + \beta - 2\pi\right] + 2\Delta(\Gamma) - l(I).$$

Clearly Identities A and B can be considered also as representation of the curvature in "linear terms".

Due to the above observation we obtain

Identity 2.3 (for arbitrary $\Gamma \in C^2$**).** *With the above defined* $C(\Gamma_i)$ *we have* $C(\Gamma) = \sum_i C(\Gamma_i)$.

On the geometry of real functions. *3.0. Synopsis.* In what follows we denote by D the domain whose boundary ∂D is a piecewise smooth curve of finite length. Denote by $\bar\Omega$ the closure of Ω. Assume that $u(x,y) \in C^1(\bar D)$ $(\in C^2(\bar D))$ and $|\mathrm{grad} u| \neq 0$ in $\bar D$. Similar functions we denote by $u(x,y) \in \tilde C^1(\bar D)$ $(\in \tilde C^2(\bar D))$ and in discussion we will refer them as *bad functions (good functions)*. The level set $\gamma(A)$ of similar functions, that is the set $\gamma(A) := \{(x,y) \in \bar D : \ u(x,y) = A\}$, $A \in R$ consists of smooth curves.

In this section we study the geometry of the level set. Particularly we give bounds for: the integral $\int_{\gamma(A)} K dl$; the length $L(\bar D, A, u)$; the absolute integral curvature $T(\bar D, A, u) = \int_{\gamma(A)} |k| dl$, where $|k|$ is the curvature of $\gamma(A)$; the cardinality $C^{\mathrm{O}}(\bar D, A, u)$ of *Hilbert's ovals*, that is the number of maximal closed connected components of $\gamma(A) \cap \bar D$; the cardinality $C_d(\bar D, A, u)$ in general case, that is the number of maximal connected components of $\gamma(A) \cap \bar D$ (both closed and non closed) which intersect $d \subset D$.

The results are connected with the integral geometry, Gamma-lines, Nevanlinna theory, Hilbert problem 16, applied topics, admit different modifications and make use of various characteristics. Moreover, in some cases we have a long lists of preceding studies. Respectively a usual presentation starting with the references and historical comments would not be optimal in this case. This is why we prefer to give first a summary of the result related to different concepts in terms of one of the characteristics. This should show demonstrably the interrelations.

Let $\bar{X}(\theta)$ be the oriented straight line passing through zero and having direction θ, that is $\bar{X}(\theta) := \{(x,y)|\ \theta := \arctan(y/x)\}$. Denote by $J_{X(\theta)}$ the oriented straight line composing the angle $\theta+\pi/2$ with x-axis and passing through the point on $\bar{X}(\theta)$ with the coordinate $X(\theta)$. We will use notation $X(\theta)$ for the coordinate on $\bar{X}(\theta)$ and $Y(\theta)$ for the coordinate on $J_{X(\theta)}$. Denote by $(X(\theta))$ the orthogonal projection of D on axis $\bar{X}(\theta)$.

For a given interval ω notation $\mathrm{Var}_\omega f$ stands for the variation of function f on ω. Denoting by β the angle formed by gradient vector of u and by positive direction of x-axis we can define now the following *rotational variational characteristic* $\mathrm{V_{Rot}}\,(D,K,u)$*:*

$$\frac{1}{\pi}\int_0^\pi \int_{(X(\theta))} \mathrm{Var}_{J_{X(\theta)}}(K\sin{(\beta-\theta)})dX(\theta)d\theta.$$

Clearly, when we consider the above concepts for a given function u and different values, say A_1, A_2,...A_q, the outcomes can be quite different but they can be also somehow interrelated. The last situation we will refer as Nevanlinna type phenomena since his known deficiency relation is of this type: $\sum_\nu \delta(a_\nu) \le 2$.

We give below several sharp inequalities in terms of $\mathrm{V_{Rot}}\,(D,K,u)$ related to the above concepts and to a given value (level) A. For each concept we present then corresponding Nevanlinna type phenomenon dealing with A_1, A_2,...A_q.

3.1. Inequalities for a given A.

Inequality 3.1 (the integral). *For any function $u(x,y) \in \tilde{C}^1(\bar{D})$, any continuous function $K(x,y)$ in $\bar{D}$ and any $A \in R$ we have*

$$\int_{\gamma(A)} K dl \le \mathrm{V_{Rot}}\,(D,K,u) + \frac{2}{\pi}\int_{\partial D}|K|dl.$$

Inequality 3.2 (the length). *For any function $u(x,y) \in \tilde{C}^1(\bar{D})$ and any $A \in R$ we have*

$$L(\bar{D},A,u) \le \mathrm{V_{Rot}}\,(D,1,u) + \frac{2}{\pi}\int_{\partial D} dl.$$

Inequality 3.3 (the absolute integral curvature). *For any function $u(x,y) \in \tilde{C}^2(\bar{D})$ and any $A \in R$ we have*

$$T(\bar{D},A,u) \le \mathrm{V_{Rot}}\,(D,|k|,u) + \frac{2}{\pi}\int_{\partial D}|k|dl,$$

where $k(x,y)$ stands for the curvature of $\gamma(A^*)$, $A^* = u(x,y)$.

Inequality 3.4 (the cardinality of the Hilbert's ovals). *For any function $u(x,y) \in \tilde{C}^2(\bar{D})$ and any $A \in R$ we have*

$$C^{\mathrm{O}}(\bar{D},A,u) \le \frac{1}{2\pi}\mathrm{V_{Rot}}\,(D,|k|,u) + \frac{1}{\pi^2}\int_{\partial D}|k|dl.$$

Inequality 3.5 (the cardinality in general case). *For any function* $u(x,y) \in \tilde{C}^2(\bar{D})$*, any domain* $d \subset D$ *with* $\Delta := \operatorname{dist}(\partial d, \partial D) > 0$ *and any* $A \in R$ *we have*

$$C_d(\bar{D}, A, u) \leq \frac{1}{2\pi} \mathrm{V}_{\mathrm{Rot}}\left(D, |k|, u\right) + \frac{1}{2\Delta} \mathrm{V}_{\mathrm{Rot}}\left(D, 1, u\right) + \frac{1}{\pi} \int_{\partial D} \left(\frac{|k|}{\pi} + \frac{1}{\Delta} \right) |dl.$$

3.2. Nevanlinna type phenomena for A_1**,** A_2**,** $...A_q$**.** *The following results are analogs of the second main theorem for Gamma-lines (which in turn is an analog of the Nevanlinna second fundamental theorem).*

Inequality 3.6 (the integral). *For any function* $u(x,y) \in c^1(\bar{D})$*, any continuous function* $K(x,y)$ *in* $\bar{D}$*, any continuous function* $\omega(t) > 0$ *on* R *and any real values* $A_1 < A_2 < ... < A_q$*,* $1 < q < \infty$*, we have*

$$\sum_{\nu=1}^{q} \int_{\gamma(A_\nu)} K dl \leq \mathrm{V}^*_{\mathrm{Rot}}\left(D, K, u\right),$$

where

$$\mathrm{V}^*_{\mathrm{Rot}}\left(D, K, u\right) := \mathrm{V}_{\mathrm{Rot}}\left(D, K, u\right) + \frac{4}{\pi\rho} \int\int_D |K| \left|\mathrm{grad}\ u\right| \omega(u) d\sigma + \frac{2}{\pi} \int_{\partial D} |K| dl,$$

$\rho = \min_\nu \int_{A_\nu}^{A_{\nu+1}} \omega(t) dt.$

Inequality 3.7 (the length). *Under the same assumptions we have*

$$\sum_{\nu=1}^{q} L(\bar{D}, A_\nu, u) \leq \mathrm{V}^*_{\mathrm{Rot}}\left(D, 1, u\right).$$

Inequality 3.8 (the absolute integral curvature). *Assuming, in addition that* $u(x,y) \in \tilde{C}^2(\bar{D})$ *we have*

$$\sum_{\nu=1}^{q} T(\bar{D}, A_\nu, u) \leq \mathrm{V}^*_{\mathrm{Rot}}\left(D, |k|, u\right).$$

Inequality 3.9 (the cardinality of the Hilbert's ovals). *Under the same assumptions we have*

$$\sum_{\nu=1}^{q} C^{\mathrm{O}}(\bar{D}, A_\nu, u) \leq \frac{1}{2\pi} \mathrm{V}^*_{\mathrm{Rot}}\left(D, |k|, u\right).$$

Inequality 3.10 (the cardinality in general case). *Under the same assumptions we have*

$$\sum_{\nu=1}^{q} C_d(\bar{D}, A_\nu, u) \leq \frac{1}{2\pi} \mathrm{V}^*_{\mathrm{Rot}}\left(D, |k|, u\right) + \frac{1}{\Delta} \mathrm{V}^*_{\mathrm{Rot}}\left(D, 1, u\right).$$

3.3. Deficiency relations. The above Nevanlinna type results admit corresponding deficiency relation. We will give just one of the versions related to the length.

Assume that a given domain D can be exhausted by $D_n \subset D$, $n \to \infty$ and $(\mathrm{V}_{\mathrm{Rot}}(D_n, 1, u) / \mathrm{V}^*_{\mathrm{Rot}}(D_n, 1, u)) \to 1$. Then defining the deficiency $\delta(A)$ of a given value (level) A as $\liminf_{n\to\infty} (L(D_n, A_\nu, u)/\mathrm{V}_{\mathrm{Rot}}(D_n, 1, u))$ we obtain from Inequality 3.7 the following

Deficiency relation for real functions:

$$\sum_{\nu=1}^{q} \delta(A_\nu) \leq 1.$$

Following Nevanlinna we may ask about examples of functions which have prescribed deficiencies for the prescribed values A_ν. We give a complete solution of this problem.

3.4. Further modifications. Each above inequality admits two modification which make use of two other characteristics (instead of rotational variational characteristics). We have not enough space to present these modifications. However, we should mention that one of these modifications (a modified version of Inequality 3.2) improves Theorem 1.1 in [4] by G. Sukiasyan.

The cardinality and integral curvature for the polynomials. Let $P(x, y)$ be a polynomial. The cardinality of the level set $\gamma(R^2, A, P) := \{(x, y) \in R^2 \backslash \infty : P(x, y) = A\}$ is of a special interest since it was widely studied in the frame of the Hilbert problem 16 [7]: "to study the number, form and positions of connected components" of the polynomials.

Related studies have a long history. Traditionally the above mentioned number "cardinality" was studied in terms of Euler's characteristics and Betty's numbers for the polynomials $P(x, y)$; see the initial, key result by Petrovskii [10], and then by Petrovskii and Oleynik [11], Thom [13], Milnor [8].

Much later the cardinality started to play an important role in computations (complexity theory by Smale). Smale and his co-authors return in [5] to the "natural or deterministic" definition of the cardinality, more convenient (visible) than the Euler's characteristics and Betty's numbers. Notice that for a given regular value A (means $|\mathrm{grad} u(x, y)| \neq 0$ on $\gamma(R^2, A, P)$) the set $\gamma(R^2, A, P)$ consists of isolated smooth curves (without intersection points in $R^2 \backslash \infty$) so that we can determine the maximal connected componenets count $\gamma_i \in \gamma(R^2, A, P)$, $i = 1, 2, ..., C(A, P) \leq \infty$ and similarly we can determine maximal closed connected components (Hilbert's ovals) o_j, $j = 1, 2, ..., C^{\mathrm{O}}(A, P)$. In these terms they prove for any regular value A the following key inequality ([5], p.303)

$$C^{\mathrm{O}}(A, P) \leq \frac{1}{2} n(P)(n(P) - 1), \qquad (*)$$

where $n(P)$ is the degree of P and derive from here for arbitrary A ([5], p.307 [1])

$$C(A,P) \le n(P)(2n(P)-1). \qquad (**)$$

Notice that (*) generalizes known Harnak's inequality which is a bit stronger but relates to the irreducible polynomials merely.

Inequality 4.1. *For any $P(x,y)$ and any regular value A we have*

$$C^{\mathrm{O}}(A,P) \le \frac{1}{2}n(P)n'(P).$$

where $n'(P) := \min[n(P'_x), n(P'_y)]$.

Inequality 4.2. *For any $P(x,y)$ and any regular value A we have*

$$C(A,P) \le \frac{1}{2}n(P)n'(P) + \pi n(P).$$

Clearly Inequality 4.1 implies (*). Moreover, when $n'(P)$ is essentially smaller than $n(P)$ Inequality 4.1 gives much more better bounds than in (*) and Inequality 4.2 gives much more better bounds than in (**).

One can easily see that Inequalities 3.5 gives far going generalization of (*) (and consequently preceding studies by Petrovskii-Oleynik-Thom-Milnor) and in addition Inequality 3.10 connects these studies with Nevanlinna type results and deficiencies.

Clearly the absolute integral curvature $T(A,P)$ of $\gamma_{R^2}(A,P)$ is a concept closely connected with this ring of problems. Surprisingly this concept has not been studied for the polynomials. We prove

Inequality 4.3. *For any $P(x,y)$ and any regular value A we have*

$$T(A,P) \le \frac{\pi}{2}\left[5n^2(P) - (6-\pi)n(P)\right].$$

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[1] On page 315 in [5] the authors assert that (*) is due to Petrovskii-Oleynik-Thom-Milnor and of course this is so substantially. However it should be stressed that (*) does not covers all the particularities we find in the mentioned works. On the other hand it seems that some details in the proofs of (*) are due to the authors of the book thus (*) should not be considered simply as an exposition.

G. Barsegian

Miscellanea in Integral Geometry, Algebraic Geometry and Real Analysis

Generalisations of some key results in integral geometry, Hilbert problem 16, Nevanlinna and Gamma-lines theories are announced.

Г. А. Барсегян

Некоторые результаты в интегральной геометрии, алгебраической геометрии и вещественном анализе

Анонсируются обобщения некоторых ключевых результатов в интегральной геометрии, в 16-й проблеме Гильберта, в теориях Неванлинны и Гамма-линий.

Գ. Ա. Բարսեղյան

Որոշ արդյունքներ ինտեգրալ երկրաչափությունում, հանրահաշվական երկրաչափությունում եւ իրական անալիզում

Բերվում են ինտեգրալ երկրաչափությունում, Հիլբերտի 16-րդ պրոբլեմում, Նեւանլիննայի եւ Գամմա-գծերի տեսություններում որոշ հիմնական արդյունքների ընդհանրացում:

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