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On a Conjecture on Lagrange Interpolation in R^3

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Introduction. The geometric characterization (GC_n) for multivariate interpolation is equivalent to the existence of a Lagrange formula with fundamental polynomials that are products of n linear factors. The Gasca-Maeztu conjecture for R^k is: for any GC_n set in R^k there is a hyperplane (called maximal hyperplane) passing through $dim\Pi_n^{k-1}$ points of that set. The conjecture for $k = 2$ has only been proved for degrees $n \leq 4$ (see e.g. [1], [2], [3]) and for $k = 3$ only for degrees $n \leq 2$ [4]. The maximal hyperplanes play important role in the study of GC_n sets. In this paper we give a precise lower estimate for the number of maximal planes for GC_2 sets in R^3.

1. Auxiliary Results. Let $\Pi_n^k := \Pi_n(R^k)$ be the space of all polynomials in k variables of total degree not exceeding n, whose dimension is $\binom{n+k}{k}$. Let us fix any finite set $X = \{\overline{x}^{(1)}, \overline{x}^{(2)}, \dots, \overline{x}^{(d)}\} \subset R^k$ as the set of knots of interpolation and pose the

Lagrange interpolation problem. Given $X = \{\overline{x}^{(1)}, \overline{x}^{(2)}, \dots, \overline{x}^{(d)}\} \subset R^k$ and any values $c_1, c_2, \dots, c_d$, find $p \in \Pi_n^k$ such that $p(\overline{x}^{(j)}) = c_j$, $j = 1, 2, \dots, d$.

Every polynomial p of degree not exceeding n can be written in the form $p(\overline{x}) = \sum\limits_{|\alpha| \leq n} a_\alpha \overline{x}^\alpha$, and the interpolation conditions give rise to the following system of d equations and $\binom{n+k}{k}$ unknowns:

$$p(\overline{x}^{(j)}) = \sum_{|\alpha| \leq n} a_\alpha (\overline{x}^{(j)})^\alpha = c_j, \quad j = 1, 2, \dots, d, \tag{1.1}$$

where a_α are the unknowns.

An interesting problem in multivariate interpolation is to infer the existence and uniqueness of the solution of the Lagrange interpolation problem from the distributions of the points in X. This leads to the following

Definition 1.1. *We say that a set $X \subset R^k$ is Π_n^k-correct, if the Lagrange interpolation problem for X and Π_n^k has a unique solution for any values $c_1, c_2, \dots, c_d$.*

From (1.1) we get that a necessary condition for a set X to be Π_n^k-correct is that $d = \begin{pmatrix} n+k \\ k \end{pmatrix}$. In that case the linear system (1.1) has the same number of equations and unknowns. Hereon we assume that this condition holds. Then the set X is Π_n^k-correct, if and only if there exists no $p \in \Pi_n^k$ vanishing at all the points of X. This condition means geometrically that no algebraic hypersurface of degree not exceeding n passes through all the points of X.

We say that $p \in \Pi_n^k$ is a fundamental polynomial or Lagrange polynomial associated to the knot $A = \overline{x}^{(i)} \in X$, if $p(\overline{x}^{(j)}) = \delta_{ij}$, $1 \le j \le d$, where δ_{ij} is the symbol of Kronecker. From now on we will denote the above fundamental polynomial by $p_A^* := p_i^*$.

Note that the set X is Π_n^k-correct if and only if all interpolation knots have fundamental polynomials. Furthermore, in the case of correctness the solution p of the Lagrange interpolation problem can be expressed by the Lagrange formula $p = \sum_{j=1}^{d} p(\overline{x}^{(j)}) \cdot p_j^*$.

Remark 1.1. *Any fundamental polynomial is of exact degree n.*

Indeed, let h be an arbitrary hyperplane $(deg h = 1)$ vanishing at A. If $deg(p_A^*) < n$, then $hp_A^* \in \Pi_n^k$ vanishes at X, contradicting the correctness of X.

Now let us consider the construction of correct sets provided by Chung and Yao.

Definition 1.2. *(See [5]) A set of knots $X \subset R^k$, $\#X = d = \begin{pmatrix} n+k \\ k \end{pmatrix}$ is said to satisfy the geometric characterization GC_n, or is a GC_n set for short, if for all $A \in X$ there exist (at most) n affine functions h_i^A, $i = 1, 2, \dots, n$, such that the union of all hyperplanes $h_i^A = 0$ contains all knots of $X \setminus \{A\}$, but not the knot A. We say that $\{h_i^A = 0 | i = 1, 2, \dots, n\}$ is the set of hyperplanes used by the knot A.*

Henceforth let us denote by h both the hyperplane and the affine function which takes part in the equation of the hyperplane.

The GC_n condition is equivalent to the existence of all fundamental polynomials in form of products of linear factors: $p_A^* = \gamma \prod_{i=1}^{n} h_i^A$, where γ is a constant. Therefore, if X satisfies the GC_n condition, then it is Π_n^k-correct. So the set of hyperplanes used by a knot must be unique, and by Remark 1.1, it contains exactly n elements.

2. Conjectures concerning GC sets. In 1982 Gasca and Maeztu made the following conjecture.

GM**-conjecture.** *(See [6]) For every GC_n set $X \subset R^2$ there is a line passing through $n+1$ nodes of X.*

A line containing $n+1$ nodes of X is called a maximal line. So far the GM-conjecture has been verified only for $n \leq 4$ (see [1], [2], [3]). Actually, the GM-conjecture states that every Chung-Yao set (GC set) is a particular case of another well-known construction, called Berzolari-Radon: there exist lines $l_0, l_1, \ldots, l_n$, such that $l_i \setminus (l_0 \cup \ldots \cup l_{i-1})$ contains exactly $n+1-i$ nodes (see e.g. [3]). Carl de Boor generalized the GM-conjecture for R^k:

GM_k**-conjecture.** *(See [7]) For every GC_n set $X \subset R^k$ there is a hyperplane passing through $dim\Pi_n^{k-1}$ knots of X.*

A hyperplane containing $dim\Pi_n^{k-1}$ knots of X is called a maximal hyperplane.

We will use the following:

Proposition 2.1. *([7]) A hyperplane is maximal if and only if it is used by all the knots not contained in it.*

Note that the case of $k=2$ was proved in [8].

In the plane, Carnicer and Gasca have strengthened the GM-conjecture by proving the following:

Theorem 2.1. *([8]) If GM-conjecture is true, then there are at least three maximal planes.*

On the basis of this result C. de Boor made the following conjecture:

CG_k**-conjecture.** *([7]) For every GC set in R^k there are at least $k+1$ maximal hyperplanes.*

He also brought a counterexample, which shows that this conjecture is not true (see [7]).

In [4] it was proved that GM_k-conjecture is true for Π_2^3. Which also shows that the following natural generalization of Theorem 2.1 for $k=3$ is not true:

- *If GM_k-conjecture is true, then there are at least $k+1$ maximal hyperplanes.*

In the next section we prove that there exist at least 3 maximal planes for any GC_2 set in R^3. The counterexample of C. de Boor (see also the example at the end of the paper) shows that this result can not be improved.

3. The maximal planes of GC_2 set in R^3. From now on let us consider the GC_2 set $X \subset R^3$, $\#X = \binom{2+3}{3} = 10$. In [4] it was proved that in this case (i.e., for $k=3$, $n=2$) the GM_k-conjecture is true. In other words there is a plane (maximal plane) passing through $dim\Pi_2^2 = 6$ knots of X. Denote this maximal plane by h_{max}. In [4] also it was mentioned that the 6 knots of the maximal plane satisfy the GC_2 condition in R^2. We will use the following result (see [4], Lemma 3.1 and Lemma

3.2):

Lemma 3.1. *Let a set of knots $X \subset R^3$ satisfy GC_n. Then the following hold:*

(i) If a line l passes through $n+1$ knots of X, then any knot not lying on l uses a plane passing through the line l.

(ii) There is no line passing through $n+2$ knots of X.

Definition 3.1. *We say that the distinct lines $l_1, l_2, \dots, l_q$ in the plane are in general position, if no two of them are parallel and no three are concurrent.*

Definition 3.2. *We say that six nodes in the plane have Δ-structure with three lines in general position if three nodes are the vertices of the triangle, formed by the lines, and the remaining three lie one by one on the lines (see Fig. 1).*

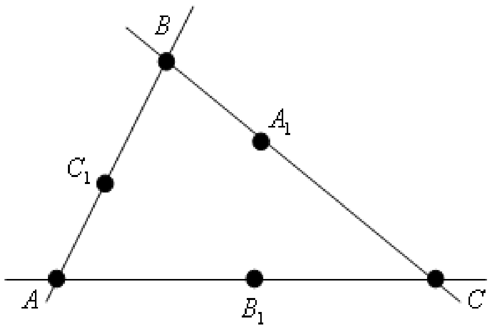

Fig. 1. Δ-structure.

We will use the following:

Proposition 3.1. *The nodes of any GC_2 set in R^2 have Δ-structure.*

Proof. Since the set of six points satisfies GC_2 in R^2 then any five of them lie on two lines. Hence there are three nodes, for example A, C_1, B, lying on a line. The remaining nodes: A_1, B_1, C are non-collinear (see Fig. 1). Consider the 2 lines used by A. It is evident that one of these lines should pass through exactly 1 node from C_1, B and 2 nodes from A_1, B_1, C. Without loss of generality assume that the line passes through the nodes B, A_1, C. Finally consider the node B. By the same way we get the third line passing through 3 nodes.$\diamondsuit$

Now we have the following distribution of 10 knots of X: 6 knots are on the plane h_{max} and have Δ-structure (by Proposition 3.1), and the remaining 4 called free knots not contained in h_{max}.

Remark 3.1. *The 4 free knots are not coplanar. In particular no three of them are collinear.*

Indeed, since otherwise the product of h_{max} and the plane, which passes through the 4 free knots, will be a non-trivial polynomial from Π_2^3 vanishing at all knots of X.

Let us fix one of three lines passing through three knots of h_{max} and denote by l^*. We have

Lemma 3.2. *Assume that there is no other maximal plane except h_{max}. Then there exists a line passing through two free knots and one of three knots of $h_{max} \setminus l^*$.*

Next, we have the following:

Proposition 3.2. *There are at least two maximal planes for any GC_2 set X in R^3.*

The proof of Proposition 3.2 is based on Lemma 3.1 and Lemma 3.2.

Now let us formulate the basic result of the paper concerning the number of maximal planes.

Theorem 3.1. *There exist at least three maximal planes for any GC_2 set X in R^3.*

The proof of Theorem 3.1 is based on Proposition 3.2 and Lemma 3.1. Also we use the fact that two maximal planes for X intersected by a line containing 3 knots. Indeed, otherwise if they have ≤ 2 common knots, then we will have all the knots on these 2 planes, which contradicts the correctness of X.

Let us mention that in the case $k = 2$ any node uses a maximal line if the GM-conjecture is true (see [9], Proposition 2). In contrast with this the counterexample of C. de Boor shows that there is a knot (namely, the knot K in the forthcoming Fig. 3) which is not using a maximal plane. In addition we have

Corollary 3.1. *For any GC_2 set in R^3 at most one knot is not using a maximal plane in it's fundamental polynomial.*

Proof: By Theorem 3.1 there are at least 3 maximal planes for any GC_2 set in R^3. Consider first the case of 4 maximal planes. Then for each knot there is a maximal plane not passing through it, since according to [7], Fact 14(v), the intersection of 4 maximal planes is empty. So by Proposition 2.1 each knot uses a maximal plane. If there are 3 maximal planes then they intersect at exactly one knot (see [7], Fact 14(iv)), which consequently is not using a maximal plane. For any other knot there is a maximal plane not passing through it. This in view of Proposition 2.1 completes the proof. $\diamondsuit$

4. Examples. For the sake of completeness let us bring two examples, which show that the result of Theorem 3.1 can not be improved. Note that these examples essentially were introduced by Carl de Boor [7] for seemingly more specific construction.

Example 1. Let us consider the following generalization of the Δ-structure for 10 knots in R^3. Suppose we have four planes in general position, i.e., any three planes intersect at exactly one point, and the four are not concurrent. Choose four knots that are the vertices of the pyramid, formed by the planes, and the remaining six knots, so called non-vertex, are lying one by one point on the lines containing the edges of the pyramid (see Fig. 2).

This set of knots is a GC_2 set in R^3. Indeed, it is easily noticed that for any non-vertex knot there are two maximal planes not passing through it and containing all other knots. Then for each fixed vertex of the pyramid there is a maximal plane not passing through it. The second plane it uses is the one passing through the remaining three knots.

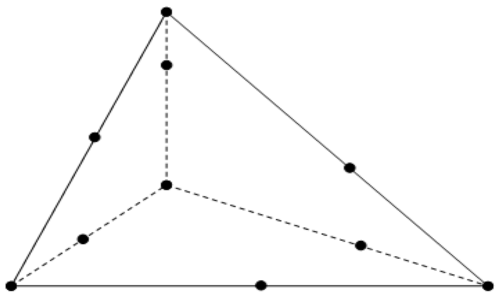

Fig. 2. A GC_2 set with 4 maximal planes.

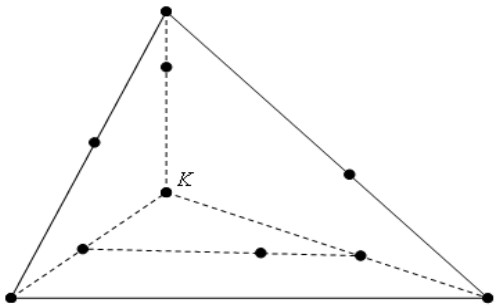

Fig. 3. A GC_2 set with 3 maximal planes.

Notice that in this example we have four maximal planes, which are the faces of the pyramid.

Example 2. Let us fix any face of the pyramid, for example the base, and move one of the non-vertex knots to the line passing through the remaining two non-vertex knots in the same face (see Fig. 3). In this case we have only three maximal planes. Besides the set of knots is also a GC_2 set in R^3. Indeed, for all knots except K there is a maximal plane not passing through it. The second plane they use is the plane passing through the remaining three knots. Finally, K uses the same planes as in the first example but containing now 4 and 5 knots, respectively.

So we get a GC_2 set in R^3, which has exactly three maximal planes. This proves that the result of Theorem 3.1 can not be improved.

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On a Conjecture on Lagrange Interpolation in R^3

The geometric characterization (GC_n) for multivariate interpolation is equivalent to the existence of a Lagrange formula with fundamental polynomials that are products of n linear factors. The Gasca-Maeztu conjecture for R^k is: for any GC_n set in R^k there is a hyperplane (called maximal hyperplane) passing through $dim \Pi_n^{k-1}$ points of that set. The maximal hyperplanes play important role in the study of GC_n sets. In this paper we give a precise lower estimate for the number of maximal planes for GC_2 sets in R^3.

Գ. Ա. Քթրյան

R^3-ում Լագրանժի միջարկման վերաբերյալ մի վարկածի մասին

Բազմաչափ միջարկման համար երկրաչափական բնութագիրը (GC_n) համարժեք է Լագրանժի բանաձևի գոյությանը, որի բոլոր ֆունդամենտալ բազմանդամները n գծային արտադրիչների արտադրյալ են: R^k-ում Գասքա-Մաեզթուի վարկածը հետևյալն է. R^k-ում ցանկացած GC_n բազմության համար գոյություն ունի հիպերհարթություն (այն կոչվում է մաքսիմալ հիպերհարթություն), որն անցնում է այդ բազմության $dim \Pi_n^{k-1}$ կետերով: Մաքսիմալ հիպերհարթությունները կարևոր դեր են խաղում GC_n բազմությունների ուսումնասիրության մեջ: Այս աշխատանքում մենք փալիս ենք R^3-ում GC_2 բազմության մաքսիմալ հարթությունների քանակի ճշգրիտ ստորին գնահատականը:

Г. А. Ктрян

О гипотезе, относящейся к интерполяции Лагранжа в R^3

Геометрическая характеристика (GC_n) для многомерной интерполяции эквивалентна существованию формулы Лагранжа, фундаментальные полиномы которой являются произведением n множителей. Гипотеза Гаска-Маезту в R^k следующая: для любого множества GC_n в R^k существует гиперплоскость (называется максимальной гиперплоскостью), которая проходит через $dim \Pi_n^{k-1}$ точек этого множества. Максимальные гиперплоскости играют важную роль в изучении множеств GC_n. Приведена точная нижняя оценка количества максимальных плоскостей для множества GC_2 в R^3.

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