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D. I. Bardzokas, G. I. Sfyris

Formulation of the State Equations for the Mixed Dynamic Problem of Elasticity for a Cracked Medium

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Key words: *theory of elasticity, cracked medium, state equations, complex potentials, mixed dynamic problem, singular integro-differential equation*

Abstract. It's common knowledge that two-dimensional and static problems of elasticity for isotropic and anisotropic multiply connected bodies may be treated with the help of Complex Potentials and the theory of singular integro-differential equations. This work is the continuation of the studies [1-3] for the cracked body; based on the framework and following the process described in [4,5] for the case of the continuous medium, we derive a system of singular integro-differential equations using properly selected Complex Potentials named $\Phi(z_1)$ and $\Psi(z_2)$. This system allows for the description of the stress and deformation field for the multiply connected and cracked body under the condition of plane strain.

1. Boundary conditions for the mixed problem and the Complex Potentials. In the frame of plane strain we examine a half-space that contains a reinforced longitudinal tunnel γ and a crack l. The displacements on the crack lips, as well as the stress at the borders $L(-\infty,\infty)$ of the half-space are considered known (Fig. 1).

On γ, which is the contour where the (there is no detachment of the reinforcement at any place) cylindrical shell with radius R (stringer) comes in contact with the tunnel hole we have the following boundary conditions, similarly to [6]:

$$\left.\begin{aligned} \frac{1}{R}\frac{dT(\theta)}{dv}+\sigma_t=0 \\ -\frac{T(\theta)}{R}+\sigma_n=0 \end{aligned}\right\}, \tag{1}$$

where

$$T(\theta) = \frac{E_0 h}{1-\nu_0^2}\varepsilon_\theta^{(str)} \tag{2}$$

is the force per unit length on the contour of the circular stringer. Also $\varepsilon_\theta = \varepsilon_\theta^{(str)} = \frac{1}{E}\left[(1-\nu^2)/\sigma_s - \nu(1+\nu)\sigma_n\right]$ is the Hooke's Law where: E_0, ν_0 and h are the elastic constants and the thickness of the stringer (cylindrical shell), E and ν are the elastic constants of the body ε_θ and ε_θ^{str} are the transverse deformations on the boundary γ of the body and the stringer reciprocally.

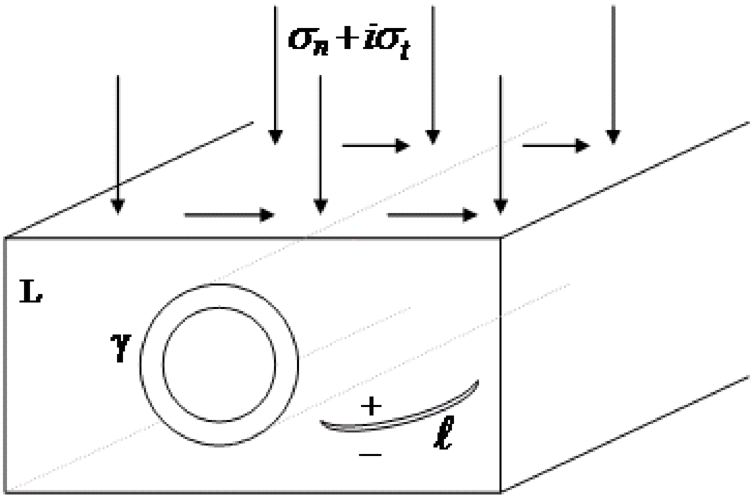

Fig. 1. A half-space that contains a reinforced longitudinal tunnel γ and a crack l. $L(-\infty,\infty)$ is the boundary of the half-space.

Combining of boundary conditions (1) and (2) produces the following equation on γ:

$$ER(1-\nu_0^2)(\sigma_n - i\sigma_t) + E_0 h(1+\nu)\left(1 - t\frac{d}{dt}\right)\left[(1-\nu)(\sigma_n+\sigma_s)-\sigma_n\right] = 0, \;\; t\in\gamma. \tag{3}$$

The basic relations of the components of the stress tensor and the vector of the displacement were used in [2,5] and can be expressed through the Complex Potentials $\Phi(z_1)$ and $\Psi(z_2)$ as follows:

$$\left.\begin{aligned} \sigma_{xx} &= -2Re\left[\left(a_1^2 + \frac{1}{2}(1-a_2^2)\right)\Phi(z_1) + \frac{1}{2}(1+a_2^2)\Psi(z_2)\right] \\ \sigma_{\psi\psi} &= (1+\alpha_1^2)Re[\Phi(z_1)+\Phi(z_2)] \\ \sigma_{x\psi} &= -2Re\left[i\left(a_1\Phi(z_1) + \frac{1+a_2^2}{4a_2}\Psi(z_2)\right)\right] \end{aligned}\right\}; \tag{4}$$

$$\left.\begin{aligned} \mu u &= -Re\left[\phi(z_1) + \frac{1}{2}(1+a_2^2)\phi(z_2)\right] \\ \mu v &= Im\left[a\phi_1(z_1) + \frac{1+a_2^2}{a_2^2}\phi(z_2)\right] \end{aligned}\right\}, \tag{5}$$

where: $\phi'(z_1) = \Phi(z_1)$ and $\psi'(z_2) - \Psi(z_2)$,

$$ia_1 = \mu_1^* = i\left[1 - \frac{\rho c^2}{\lambda + 2\mu}\right]^{1/2} = i\left[1 - \frac{c^2}{c_1^2}\right],\ ia_2 = \mu_2^* = i\left[1 - \frac{\rho c^2}{\mu}\right]^{1/2} = i\left[1 - \frac{c^2}{c_2^2}\right],$$

μ_1^* and μ_2^* are the solutions of the characteristic equation

$$\left[1 - \frac{\rho c^2}{\lambda + 2\mu} + (\mu^*)^2\right]\left[1 - \frac{\rho c^2}{\mu} + (\mu^*)^2\right] = 0,$$

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}},\ c_2 = \sqrt{\frac{\mu}{\rho}},\ z_j = x + \mu_j^* y,\ j = 1, 2,$$

c_1 and c_2 are the speeds of the primary (P) and secondary (S) waves in the medium. c is the speed of the distortion along x-axis and λ, μ are the Lame constants.

We will express the Complex Potentials $\Phi(z_1)$ and $\Psi(z_2)$ using the Cauchy integrals just like the case of the anisotropic medium [1,2] as follows:

$$\Phi(z_1) = \frac{1}{2\pi i}\int\limits_L \frac{\phi_0(\tau_1)}{\tau_1 - z_1}d\tau_1 + \frac{1}{2\pi i}\oint\limits_{\gamma_1} \frac{\mu(\tau_1)}{\tau_1 - z_1}d\tau_1 + \frac{1}{2\pi i}\int\limits_{l_1} \frac{\phi(\tau_1)}{\tau_1 - z_1}d\tau_1; \tag{6}$$

$$\Psi(z_2) = \frac{1}{2\pi i}\int\limits_L \frac{\psi_0(\tau_2)}{\tau_2 - z_2}d\tau_2 + \frac{1}{2\pi i}\oint\limits_{\gamma} \frac{w(\tau_2)}{\tau_2 - z_2}d\tau_2 + \frac{1}{2\pi i}\int\limits_{l_2} \frac{\psi(\tau_2)}{\tau_2 - z_2}d\tau_2. \tag{7}$$

2. Formation of the State Equations. We continue in a similar way with the works [4,5] where the first and second fundamental problems where examined, and based on (4) and (5) we write the stresses and displacements with respect to $\Phi(z_1)$ and $\Psi(z_2)$ on the boundary of the body. On $L(-\infty, \infty)$ we have:

$$\begin{aligned} &2[\sigma_n + i\sigma_t] = (\sigma_{xx} + \sigma_{\psi\psi}) - e^{-2i\theta}[\sigma_{xx} - \sigma_{\psi\psi} + 2i\sigma_{x\psi}] = \\ &= -(a_1^2 - a_2^2)[\Phi(t_1) + \overline{\Phi(t_1)}] + \frac{\overline{dt}}{dt}\{(1 + a_1^2)(\Phi(t_1) + \overline{\Phi(t_1)}) + \\ &+ (1 + a_2^2)(\Psi(t_2) + \overline{\Psi(t_2)}) + 2i\{a_1[i\Phi(t_1) - i\overline{\Phi(t_1)}] + \frac{(1 + a_2^2)}{4a_2}[i\Psi(t_2) - i\overline{\psi(t_2)}]\}\}, \end{aligned} \tag{8}$$

$$\frac{dt}{\overline{\overline{dt}}} = e^{-2i\theta}, \quad t \in \gamma,$$

where:

$$\begin{aligned} &\Phi(t_1) = \frac{1}{\pi i}\int\limits_{L_1} \frac{\phi_0(\tau_1)}{\tau_1 - t_1}d\tau_1 + \frac{1}{2\pi i}\oint\limits_{\gamma_1} \frac{\mu(\tau_1)}{\tau_1 - t_1}d\tau_1 + \frac{1}{2\pi i}\int\limits_{l_1} \frac{\phi(\tau_1)}{\tau_1 - t_1}d\tau_1, \\ &\Psi(t_2) = \frac{1}{\pi i}\int\limits_{L_2} \frac{\Psi_0(\tau_2)}{\tau_2 - t_2}d\tau_2 + \frac{1}{2\pi i}\oint\limits_{\gamma_2} \frac{w(\tau_1)}{\tau_2 - t_2}d\tau_2 + \frac{1}{2\pi i}\int\limits_{l_2} \frac{y(\tau_2)}{\tau_2 - t_2}d\tau_2, \quad t_1, t_2 \in L_1, L_2, \end{aligned} \tag{9}$$

on l we have:

$$2\mu\frac{d}{dt}[u^{\pm}(t)+iv^{\pm}(t)] = -[\beta_1\frac{dt_1}{dt}\Phi^{\pm}(t_1)+\beta_2\overline{\frac{dt_1}{dt}}\overline{\Phi^{\pm}(t_1)}+ \\ +\beta_3\frac{dt_2}{dt}\Psi^{\pm}(t_2)+\beta_4\overline{\frac{dt_2}{dt}}\overline{\Psi^{\pm}(t_2)}], \tag{10}$$

where:

$$\beta_1 = 1+\alpha_1,\ \beta_2 = 1-\alpha_1,\ \beta_3 = \frac{1}{2}(1+\alpha_2^2)\left(1+\frac{1}{\alpha_2}\right),\ \beta_4 = \frac{1}{2}(1+\alpha_2^2)\left(1-\frac{1}{\alpha_2}\right),$$

$$\Phi^{\pm}(t_1) = \pm\frac{1}{2}\phi(t_1)+\frac{1}{2\pi i}\int\limits_{l_1}\frac{\phi(\tau_1)}{\tau_1-t_1}d\tau_1+\frac{1}{2\pi i}\int\limits_{L_1}\frac{\phi_0(\tau_1)}{\tau_1-t_1}d\tau_1+\frac{1}{2\pi i}\oint\limits_{\gamma_1}\frac{w(\tau_1)}{\tau_1-t_1}d\tau_1,$$

$$\Psi^{\pm}(t_2) = \pm\frac{1}{2}y(t_2)+\frac{1}{2\pi i}\int\limits_{l_2}\frac{y(\tau_2)}{\tau_2-t_2}d\tau_2+\frac{1}{2\pi i}\int\limits_{L_2}\frac{\psi_0(\tau_2)}{\tau_2-t_2}d\tau_2+\frac{1}{2\pi i}\oint\limits_{\gamma_2}\frac{w(\tau_2)}{\tau_2-t_2}d\tau_2, t_1,t_2\in l_1,l_2,$$

$$\frac{dt_j}{dt} = \frac{1}{2}\left[(1+a_j)+(1-a_j)\frac{\overline{dt}}{dt}\right],\ \frac{dt_j}{\overline{dt}} = \frac{1}{2}\left[(1+a_j)\frac{dt}{\overline{dt}}+(1-a_j)\right],\ j=1,2. \tag{11}$$

On γ based on (3) and (4) we get:

$$ER(1-\nu^2)\Big\{-(a_1^2-a_2^2)[\Phi(t_1)+\overline{\Phi(t_1)}]+\frac{dt}{\overline{dt}}\Big\{(1+a_1^2)(\Phi(t_1)+\overline{\Phi(t_1)})+ \\ +(1+a_2^2)(\Psi(t_2)+\overline{\Psi(t_2)})-2i\{a_1[i\Phi(t_1)-i\overline{\Phi(t_1)}]+ \\ +\frac{(1+a_2^2)}{4a_2}[i\Psi(t_2)-i\overline{\Psi(t_2)}]\}\}\Big\}+2E_0L(1+\nu)\left(1-t\frac{d}{dt}\right)\times \\ \times\{\nu(a_1^2-a_2^2)(\Phi(t_1)+\overline{\Phi(t_1)})-Re\frac{dt}{\overline{dt}}\{(1+a_1^2)(\Phi(t_1)+\overline{\Phi(t_1)})+ \\ +(1+a_2^2)(\Psi(t_2)+\overline{\Psi(t_2)})\}\}=0, \tag{12}$$

where:

$$\Phi(t_1) = \frac{1}{\pi i}\oint\limits_{\gamma_1}\frac{\mu(\tau_1)}{\tau_1-t_1}d\tau_1+\frac{1}{2\pi i}\int\limits_{L}\frac{\phi_0(\tau_1)}{\tau_1-t_1}d\tau_1+\frac{1}{2\pi i}\int\limits_{l_1}\frac{\phi(\tau_1)}{\tau_1-t_1}d\tau_1, \\ \Psi(t_2) = \frac{1}{\pi i}\oint\limits_{\gamma_2}\frac{w(\tau_2)}{\tau_2-t_2}d\tau_2+\frac{1}{2\pi i}\int\limits_{L}\frac{\psi_0(\tau_2)}{\tau_2-t_2}d\tau_2+\frac{1}{2\pi i}\int\limits_{l_2}\frac{y(\tau_2)}{\tau_2-t_2}d\tau_2,\ t_1,t_2\in\gamma_1,\gamma_2. \tag{13}$$

The boundary conditions on L,γ and l through relations (8) and (10) and the Sohotsky - Plemely formulas allows us to express the complex potential $\Psi(z_2)$ as function of $\phi_0(t_1)$, $\mu(t_1)$ and $\phi(t_1)$ in the following way:

$$\Psi(z_2) = -\frac{1}{2\pi i}\int\limits_{L_1}\frac{\lambda_0(\tau)}{\tau_2-z_2}d\tau-\frac{1}{2\pi i}\int\limits_{L_1}\frac{A_0(\tau,\overline{\tau})}{\tau_2-z_2}\phi_0(\tau_1)d\tau_1-\frac{1}{2\pi i}\int\limits_{L_1}\frac{B_0(\tau,\overline{\tau})}{\tau_2-z_2}\overline{\phi_0(\tau_1)}d\tau_1-$$

$$
-\frac{\beta_1\beta_3-\beta_4\beta_2}{\beta_3^2-\beta_4^2}\frac{1}{2\pi i}\oint_{\gamma_1}\frac{\mu(\tau_1)}{\tau_2-z_2}d\tau_1-\frac{\beta_3\beta_2-\beta_1\beta_4}{\beta_3^2-\beta_4^2}\frac{1}{2\pi i}\oint_{\gamma_1}\frac{\overline{\mu(\tau_1)}}{\tau_2-z_2}\overline{d\tau_1}- \tag{14}
$$

$$
-\frac{1}{2\pi i}\int_{l_1}\frac{\lambda_1(\tau)}{\tau_2-t_2}d\tau-\frac{\beta_1\beta_3-\beta_4\beta_2}{\beta_3^2-\beta_4^2}\frac{1}{2\pi i}\int_{l_1}\frac{\Phi(\tau_1)}{\tau_2-z_2}d\tau_1-\frac{\beta_3\beta_2-\beta_1\beta_4}{\beta_3^2-\beta_4^2}\frac{1}{2\pi i}\int_{l_1}\frac{\overline{\phi(\tau_1)}}{\tau_2-z_2}\overline{d\tau_1},
$$

where:

$$
\begin{aligned}
&\lambda_0(t)=\left[\frac{2a_2}{(1+a_2^2)(1+a_2)^2}\frac{dt}{\overline{dt}}(\sigma_n+i\sigma_t)+\right.\\
&\left.+\frac{2a_2}{(1+a_2^2)(1+a_2)^2}\frac{\overline{dt}}{dt}(\sigma_n-i\sigma_t)\right]\frac{(1-a_2^2)^2}{8a(1+a_2^2)^2}\frac{dt_2}{dt},\\
&A_0(t,\overline{t})=\frac{(1-a_2^2)^2}{4(1+a_2^2)^2}\left\{\frac{1}{(1+a_2)^2}\left[(1-a_1)^2+\frac{dt}{\overline{dt}}(a_2^2-a_1^2)\right]+\right.\\
&\left.+\frac{1}{(1-a_2)^2}\left[(1+a_1)^2+\frac{\overline{dt}}{dt}(a_2^2-a_1^2)\right]\right\}\frac{dt_2}{dt_1},\\
&B_0(t,\overline{t})=\frac{(1-a_2^2)^2}{4(1+a_2^2)^2}\left\{\frac{1}{(1+a_2)^2}\left[(1+a_1)^2+\frac{dt}{\overline{dt}}(a_2^2-a_1^2)\right]+\right.\\
&\left.+\frac{1}{(1-a_2)^2}\left[(1-a_1)^2+\frac{\overline{dt}}{dt}(a_2^2-a_1^2)\right]\right\}\frac{dt_2}{dt_1},\\
&\lambda_1(t)=2\mu\frac{d}{dt}[(u^+-u^-)+i(v^+-v^-)]\frac{1}{\beta_3+\beta_4}.
\end{aligned} \tag{15}
$$

By substitution of the boundary relations (6) and (14) in (8), (12) and in the outcome of the difference between equations (10), based on the Sohotsky - Plemely formulas, we shall get a system of integro-differential equations regarding the densities $\phi_0(t_1)$, $\mu(t_1)$ and $\phi(t_1)$. This system is the basis for the solution of the above mentioned mixed boundary problem.

3. Conclusions. In this work a mixed dynamic problem was treated for the first time, in the framework of plane strain, for a multiply connected and cracked body. The solution of this problem is of great importance for scientific fields such as Composite Materials and Earthquake Mechanics.

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National Technical University of Athens

D. I. Bardzokas, G. I. Sfyris

Formulation of the State Equations for the Mixed Dynamic Problem of Elasticity for a Cracked Medium

We derive a system of singular integro-differential equations using properly selected Complex Potentials named $\Phi(z_1)$ and $\Psi(z_2)$. This system allows for the description of the stress and deformation field for the multiply connected and cracked body under the condition of plane strain.

Д. И. Бардзокас, Г. И. Сфирис

Формулировка определяющих уравнений смешанной динамической задачи теории упругости для сред с трещиной

Надлежащим выбором комплексных потенциалов $\Phi(z_1)$ и $\Psi(z_2)$ выводится система сингулярных интегродифференциальных уравнений одной смешанной динамической задачи теории упругости. Эта ситема описывает поле напряжений и деформаций в многосвязанном теле с трещиной при условиях плоской деформации.

Դ. Ի. Բարձոկաս, Գ. Ի. Սֆիրիս

Առաձգականության տեսության դինամիկական խառը խնդրի որոշիչ հավասարումների ձևակերպումը ճաքով միջավայրի համար

Հոդվածում $\Phi(z_1)$ եւ $\Psi(z_2)$ կոմպլեքս պոտենցիալների պատշաճ ընտրությամբ արտածվում է առաձգականության տեսության դինամիկական խառը խնդրի ինտեգրադիֆերենցիալ հավասարումների համակարգը: Այս համակարգը նկարագրում է լարումների եւ դեֆորմացիաների դաշտը ճաքով թուլացված բազմակապ տիրույթում հարթ դեֆորմացիայի պայմանների դեպքում:

References

1. *Muskhelisvili N.I.* Some basic problems of the mathematical theory of elasticity. P. Noordhoff, Groningen. 1965.

2. *Lechnitsky S.G.* Theory of elasticity for the anisotropic media. Moscow. 1951 (in Russian).

3. *Sneddon J.N., Berry D.S.* The classical theory of elasticity. Springer - Verlang, Berlin - Goetingen - Heidelberg. 1958.

4. *Bardzokas D.I., Sfyris G.I.* The state equations for the first and second fundamental problems of elastodynamics for a cracked medium. (Accepted in congress 4^{th} ICNDT of HSNT)

5. *Sfyris G.I.* Mathematical simulation in problems of coupled fields and earthquake mechanics. PhD Thesis, National Technical University of Athens, 2007 (in Greek).

6. *Alexandrov V.M., Mkhitaryan S.M.* Contact problems of bodies with thin coatings and interlayers. M. Nauka. 1987 (in Russian).