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On Distance in Variation for Frequency Distributions Generated by Stable Laws

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0. Introduction. In [1] the properties and the reasons of usefulness of *Stable Laws* in bioinformatics were explained. Namely, the following *two-parametric* families of densities concentrated on $R^+ = (0, +\infty)$ and generated by *standard* stable densities

$$s(x;\alpha) = \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\cdot\frac{1}{x^{n\alpha+1}}\cdot\sin(\pi n\alpha),\ 0<\alpha<1, \tag{0.1}$$

and

$$s(x;\alpha) = \frac{1}{\pi}\sum_{m\geq 1}(-1)^{m-1}\frac{\Gamma(\frac{2m-1}{\alpha}+1)}{(2m-1)!}\cdot x^{2m-2},\ 1<\alpha<2, \tag{0.2}$$

were introduced:

$$\{\hat{f}_{\alpha,\sigma}(x) = \sigma^{-1/\alpha}\cdot s(x\cdot\sigma^{-1/\alpha};\alpha) : 0<\alpha<1, \sigma\in R^+\}, \tag{0.3}$$

$$\{\hat{f}_{\alpha,\sigma}(x) = 2\sigma^{-1/\alpha}\cdot s(x\cdot\sigma^{-1/\alpha};\alpha) : 1<\alpha<2, \sigma\in R^+\}. \tag{0.4}$$

Here $\Gamma(\bullet)$ is the *Euler's Gamma Function.*

They were suggested as continuous analogs of empirical frequency distributions arising in large-scale biomolecular models. It was also shown that densities

$\hat{f}_{\alpha,\sigma}(x)$ satisfy *statistical facts*[1] on empirical frequency distributions and the *Integral Representations* for corresponding distribution functions (DF) were given as follows:

$$\hat{F}_{\alpha,1}(x) = \frac{1}{\pi}\int\limits_0^{\pi} \exp(-\frac{1}{x^{\alpha/(\alpha-1)}} \cdot \overline{U}_\alpha(\varphi))d\varphi, \text{ for } 0 < \alpha < 1, \tag{0.5}$$

with

$$\overline{U}_\alpha(y) = \left(\frac{\sin(\alpha\varphi)}{\sin\varphi}\right)^{\frac{\alpha}{\alpha-1}} \cdot \frac{\sin((1-\alpha)\cdot\varphi)}{\cos\varphi}, \ \varphi \in [0,\pi], \tag{0.6}$$

and

$$\hat{F}_{\alpha,1}(x) = 1 - \frac{2}{\pi}\int\limits_0^{\pi/2} \exp(-x^{\alpha/(\alpha-1)} \cdot \overline{V}_\alpha(\varphi))d\varphi, \text{ for } 1 < \alpha < 2, \tag{0.7}$$

with

$$\overline{V}_\alpha(y) = \left(\frac{\cos\varphi}{\sin(\alpha\varphi)}\right)^{\frac{\alpha}{\alpha-1}} \cdot \frac{\cos((\alpha-1)\cdot\varphi)}{\cos\varphi}, \ \varphi \in [0,\pi/2]. \tag{0.8}$$

In [1] the construction of frequency distributions based on *discretization* [2] of densities (0.3)-(0.4) which also conserve statistical facts 1-4 has been done.

In order to use any frequency distribution in bioinformatics (besides of statistical facts 1-4) one needs to verify that this distribution is *stable*, in particular, *by parameters*. It is possible to suggest various types of stability by parameters.

1. Problem and results. We deal with the following problem for continuous analogs (0.3)-(0.4). Consider the families of DF of continuous analogs

$$\{\hat{F}_{\alpha,\sigma}(x) : 0 < \alpha < 1, \sigma \in R^+\} \tag{1.1}$$

and

$$\{\hat{F}_{\alpha,\sigma}(x) : 1 < \alpha < 2, \sigma \in R^+\}. \tag{1.2}$$

Denote

$$E(\alpha,\sigma;\alpha',\sigma') = E(\alpha',\sigma';\alpha,\sigma) = \int\limits_{-0}^{+\infty} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx, \tag{1.3}$$

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[1] Based on huge datasets of great number of large-scale biomolecular system some common statistical facts for empirical frequency distributions, say $\{p_n\}$, arising in biomolecular models were postulated. Here they are.

1. $\{p_n\}$ *has a skew to the right.*
2. $\{p_n\}$ *shows a Power Law behavior as* $n \to \infty$.
3. $\{p_n\}$ *satisfies some convexity properties.*
4. $\{p_n\}$ *is unimodal.*

The fact 2 is represented in [2] as a regular variation of $\{p_n\}$ at infinity.

[2] The distribution

$$p_n = \int\limits_{n-1}^{n} f(x)dx, \ n = 1, 2, \ldots$$

is called a *discretization* of density $f(x)$.

where $\sigma \in R^+$, $\sigma' \in R^+$ and either $\alpha \in (0,1)$, $\alpha' \in (0,1)$, or $\alpha \in (1,2)$, $\alpha' \in (1,2)$.

Let the constants $\underline{\sigma}$ and $\overline{\sigma}$ be fixed and satisfy inequalities $0 < \underline{\sigma} < \overline{\sigma} < +\infty$. The constants $\underline{\alpha}$ and $\overline{\alpha}$ are fixed too and for the family of DF (1.1) satisfy the inequalities $0 < \underline{\alpha} \leq \overline{\alpha} < 1$. For the family of DF (1.2) these constants satisfy the inequalities $1 < \underline{\alpha} \leq \overline{\alpha} < 2$.

The general *Problem of Stability by Distance in Variation* for families (0.3)-(0.4) is solved in

Theorem 1. *Uniformly on* $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, $\sigma' \in [\underline{\sigma}, \overline{\sigma}]$ *the limit exists*

$$\lim_{|\alpha-\alpha'|+|\sigma-\sigma'|\to 0} E(\alpha,\sigma;\alpha',\sigma') = 0 \tag{1.4}$$

Denote

$$E_1(\alpha;\sigma,\sigma') = E_1(\alpha;\sigma';\sigma) = \int\limits_{0-}^{+\infty} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma'}(x)|dx (= E(\alpha,\sigma;\alpha,\sigma')), \tag{1.5}$$

$$E_2(\sigma;\alpha,\alpha') = E_2(\sigma;\alpha',\alpha) = \int\limits_{0-}^{+\infty} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma'}(x)|dx (= E(\alpha,\sigma;\alpha',\sigma)). \tag{1.6}$$

According to the inequality $0 \leq E(\alpha,\sigma;\alpha',\sigma') \leq E_1(\alpha;\sigma,\sigma') + E_2(\sigma';\alpha,\alpha')$, we may formulate

Remark 1. *In order to prove the statement of Theorem 1 it is enough to do it in two particular cases. Namely, uniformly on* $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, $\sigma' \in [\underline{\sigma}, \overline{\sigma}]$ *simultaneously the limits exist*

$$\lim_{|\sigma-\sigma'|\to 0} E_1(\alpha;\sigma,\sigma') = 0, \tag{1.7}$$

and

$$\lim_{|\alpha-alpha'|\to 0} E_2(\sigma;\alpha,\alpha') = 0. \tag{1.8}$$

Denote

$$\varepsilon(\alpha,\sigma;\alpha',\sigma') = \sum_{k=1}^{\infty} |(p_k(\alpha,\sigma) - p_k(\alpha',\sigma'))|, \tag{1.9}$$

where $\{p_n(\alpha,\sigma)\}$ is the discretization of $\hat{f}_{\alpha,\sigma}(x)$ and $\alpha,\sigma,\alpha',\sigma'$ are as in (1.3).

For *discretizations* of families (0.3) and (0.4) the problem of Stability by Distance in Variation consists in discovering the conditions on parameters under which the functional $\varepsilon(\alpha,\sigma;\alpha',\sigma')$ converges *uniformly* by parameters to zero. According to (1.9), (1.4) we have

$$0 \leq \varepsilon(\alpha,\sigma;\alpha',\sigma') = \sum_{k=1}^{\infty} |(p_k(\alpha,\sigma) - p_k(\alpha',\sigma'))| \leq \int\limits_{0-}^{+\infty} |\hat{f}_{\alpha,\sigma}(u) - \hat{f}_{\alpha',\sigma'}(u)|du \to 0.$$

as $|\alpha-\alpha'|+|\sigma-\sigma'|\to 0$. So, we may formulate

Theorem 2. *Uniformly on* $\alpha\in[\underline{\alpha},\overline{\alpha}]$, $\alpha'\in[\underline{\alpha},\overline{\alpha}]$, $\sigma\in[\underline{\sigma},\overline{\sigma}]$, $\sigma'\in[\underline{\sigma},\overline{\sigma}]$ *the limit exists*

$$\lim_{|\alpha-\alpha'|+|\sigma-\sigma'|\to 0}\varepsilon(\alpha,\sigma;\alpha',\sigma')=0.$$

2. Preliminary estimations I.

Lemma 1. *Given* $\varepsilon\in(0,1)$ *in conditions of Theorem 1 for DF of continuous analogs there is a number* $x_0\in R^+$ *(*x_0 *doesn't depend on* α *and* σ*) such that for all* $x\in[+\infty)$

$$1-\hat{F}_{\alpha,\sigma}(x)<\frac{\varepsilon}{16}. \tag{2.1}$$

Proof. Consider the case $0<\alpha<1$. We deal with the following series expansion ([3], p.p. 108-109)

$$1-\hat{F}_{\alpha,\sigma}(x)=\frac{1}{\pi\alpha}\sum_{n\geq 1}(-1)^{n-1}\cdot\frac{\Gamma(n\alpha+1)}{n!}\sin(\pi n\alpha)\cdot\frac{1}{x^{n\alpha}}. \tag{2.2}$$

We use the asymptotic formula ([4], p.937)

$$\Gamma(x)\sim x^{x-(1/2)}\cdot e^{-x}\cdot\sqrt{2\pi},\ x\to\infty. \tag{2.3}$$

By (2.3), $\dfrac{\Gamma(n\overline{\alpha}+1)}{n\cdot n!}\sim\dfrac{\alpha^{1/2}}{n}\cdot\left(\dfrac{e}{n}\right)^{n(1-\overline{\alpha})}\cdot\overline{\alpha}^{\overline{\alpha}\cdot n}$, $n\to\infty$ which proves the convergence of series $\dfrac{1}{n\underline{\alpha}}\sum n\geq 1\dfrac{\Gamma(n\overline{\alpha}+1)}{n\cdot n!}$. That is why for a given $\varepsilon\in(0,1)$ there is an integer $n_0>1$ such that

$$\frac{1}{n\underline{\alpha}}\sum n\geq 1\frac{\Gamma(n\overline{\alpha}+1)}{n\cdot n!}<\frac{\varepsilon}{16} \tag{2.4}$$

But for any $x\in(\max(1,(\overline{\sigma})^{1/\alpha}),+\infty)$ from (2.2) we have

$$0\leq 1-\hat{F}_{\alpha,\sigma}(x)\leq\frac{1}{\pi\underline{\alpha}}\sum n\geq 1\frac{\Gamma(n\overline{\alpha}+1)}{n\cdot n!}\cdot\frac{1}{x^{n\underline{\alpha}}}<\frac{1}{\pi\underline{\alpha}}\sum n\geq 1\frac{\Gamma(n\overline{\alpha}+1)}{n\cdot n!},$$

which, due to (2.4) proves Lemma 1 in this case.

Consider the case $1<\alpha<2$. Now, we deal with *Integral Representation* (0.7), i.e.

$$1-\hat{F}_{\alpha,\sigma}(x)=\frac{2}{\pi}\int_0^{\pi/2}\exp(-(x^{\alpha}/\sigma)^{1/(\alpha-1)}\cdot\overline{V}_{\alpha}(\varphi))d\varphi, \tag{2.5}$$

where, due to (0.8), the function $\overline{V}_{\alpha}(\varphi)$ by φ in $(0,\pi/2)$ is positive. Moreover, $\overline{V}_{\alpha}(\varphi)$ for fixed φ *decreases* as α increases.

Indeed, the functions $\dfrac{1}{\sin(\alpha\varphi)}$, $\cos((\alpha-1)\varphi)$, $\dfrac{\alpha}{\alpha-1}\left(=\dfrac{1}{1-(1/\alpha)}\right)$ decrease as α increases. So, the function $\left(\dfrac{1}{\sin(\alpha\varphi)}\right)^{\alpha/(\alpha-1)}$ decreases because the last function is more than one for $1<\alpha<2$ and $\varphi\in(0,\pi/2)$.

Since the function $\overline{V}_\alpha(\varphi)$ is positive and decreases, therefore from (2.5) for $x \in (1, +\infty)$ we obtain

$$0 \le 1 - \hat{F}_{\alpha,\sigma}(x) \le \frac{2}{\pi} \int\limits_0^{\pi/2} \exp(-(x^\alpha/\overline{\sigma})^{1/(\alpha-1)} \cdot \overline{V}_\alpha(\varphi)) d\varphi \le$$

$$\le \begin{cases} \frac{2}{\pi} \int\limits_0^{\pi/2} \exp(-x^{\overline{\alpha}/(\overline{\alpha}-1)} \cdot (\overline{\sigma})^{-1/(\underline{\alpha}-1)} \cdot \overline{V}_{\overline{\alpha}}(\varphi)) d\varphi, \ if \ \overline{\sigma} \in (0,1), \\ \frac{2}{\pi} \int\limits_0^{\pi/2} \exp(-x^{\overline{\alpha}/(\overline{\alpha}-1)} \cdot (\overline{\sigma})^{-1/(\overline{\alpha}-1)} \cdot \overline{V}_{\overline{\alpha}}(\varphi)) d\varphi, \ if \ \overline{\sigma} \in (1,+\infty). \end{cases}$$

In case $\overline{\sigma} \in (0,1)$ we solve the equation $(\hat{\sigma})^{1/(\overline{\alpha}-1)} = (\overline{\sigma})^{1/(\underline{\alpha}-1)}$ with unknown $\hat{\sigma} \in R^+$: i.e. $\hat{\sigma} = (\overline{\sigma})^{(\overline{\alpha}-1)(\underline{\alpha}-1)}$, and put

$$\sigma_* = \begin{cases} \hat{\sigma} \quad if \quad \overline{\sigma} \in (0,1), \\ \overline{\sigma} \quad if \quad \overline{\sigma} \in (1,+\infty), \end{cases} \qquad with \quad \sigma_* \in R^+$$

Thus, $1 - \hat{F}_{\alpha,\sigma}(x) \le \frac{2}{\pi} \int\limits_0^{\pi/2} \exp(-(x^\alpha/\sigma_*)^{1/(\alpha-1)} \cdot \overline{V}_\alpha(\varphi)) d\varphi$, for $x \in (1, +\infty)$ or, due to (2.5),

$$1 - \hat{F}_{\alpha,\sigma}(x) \le 1 - \hat{F}_{\alpha,\sigma_*}(x) \quad \text{for} \quad x \in (1,+\infty). \tag{2.6}$$

Since the function $1 - \hat{F}_{\alpha,\sigma_*}(x)$ *varies regularly* at infinity with exponent $(-\alpha)$, therefore for a given $\varepsilon \in (0,1)$ there is a number $x_0 \in R^+$ such that $1 - \hat{F}_{\alpha,\sigma_*}(x) < \frac{\varepsilon}{16}$ for all $x \in (x_0, +\infty)$, which together with (2.6) and $x \in (1, +\infty)$ prove (2.1) in this case.

3. Preliminary estimations II. Denote

$$\gamma(\alpha, \sigma, \alpha', \sigma') = |\hat{f}_{\alpha,\sigma}(0) - \hat{f}_{\alpha',\sigma'}(0)|, \tag{3.1}$$

According to (0.3)-(0.4) in case $1 < \alpha < 2$

$$\hat{f}_{\alpha,\sigma}(0) = \sigma^{-1/\alpha} \cdot \frac{2}{\pi} \Gamma(1 + \frac{1}{\alpha}), \tag{3.2}$$

and in case $0 < \alpha < 1$, obviously, $\hat{f}_{\alpha,\sigma}(0) = 0$.

Lemma 2. *In conditions of Theorem 1 for a given* $\varepsilon \in (0,1)$ *uniformly on* $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, $\sigma' \in [\underline{\sigma}, \overline{\sigma}]$ *the inequalities hold* $0 \le \overline{\lim\limits_{|\alpha-\alpha'|+|\sigma-\sigma'| \to 0}} \gamma(\alpha, \sigma, \alpha', \sigma') < \frac{\varepsilon}{16}$. *This limit relationship implies that for* $|\alpha - \alpha'| + |\sigma - \sigma'|$ *small enough*

$$\gamma(\alpha, \sigma, \alpha', \sigma') < \frac{\varepsilon}{8} \tag{3.3}$$

Proof. The case $0 < \alpha < 1$ is obvious. Consider the case $1 < \alpha < 2$. Due to, (3.2), we have

$$\frac{\pi}{2}\gamma(\alpha,\sigma,\alpha',\sigma') = \frac{1}{(\sigma\cdot\sigma')^{1/\alpha}}\cdot|\sigma^{1/\alpha}\cdot\Gamma(1+\frac{1}{\alpha}) - (\sigma')^{1/\alpha}\cdot\Gamma(1+\frac{1}{\alpha'})| \leq \\ \leq A_1\cdot|(\sigma')^{1/\alpha} - \sigma^{1/\alpha}| + A_2\cdot|\Gamma(1+\frac{1}{\alpha}) - \Gamma(1+\frac{1}{\alpha'})|. \tag{3.4}$$

where $A_1 = (\underline{\sigma})^{-2/\overline{\sigma}}\cdot\Gamma(1+\frac{1}{\underline{\alpha}})$, $A_2 = (\overline{\sigma})^{1/\underline{\alpha}}$. Here the monotonity of *Gamma Function* was used. Next, we have: *uniformly* on $\alpha\in[\underline{\alpha},\overline{\alpha}]$, $\alpha'\in[\underline{\alpha},\overline{\alpha}]$

$$\lim_{|\alpha-\alpha'|\to 0}|\Gamma(1+\frac{1}{\alpha}) - \Gamma(1+\frac{1}{\alpha'}) = 0.| \tag{3.5}$$

Similarly, *uniformly* on $\alpha\in[\underline{\alpha},\overline{\alpha}]$, $\sigma\in[\underline{\sigma},\overline{\sigma}]$, $\sigma'\in[\underline{\sigma},\overline{\sigma}]$,

$$\lim_{|\sigma-\sigma'|\to 0}|(\sigma')^{1/\alpha} - \sigma^{1/\alpha}| = 0. \tag{3.6}$$

The relations (3.4)-(3.6) imply the inequalities in Lemma 2.

For $\tau\in(0,1)$ denote

$$I_\tau(\alpha,\sigma) = \int_{0-}^{\tau}|(\hat{f}_{\alpha,\sigma}(0) - \hat{f}_{\alpha,\sigma}(u))|du. \tag{3.7}$$

Lemma 3. *1. There is a constant $B\in R^+$ such that uniformly on $\alpha\in[\underline{\alpha},\overline{\alpha}]$, $\sigma\in[\underline{\sigma},\overline{\sigma}]$ for all $x\in R^+$ the inequality holds*

$$\left|\frac{d}{dx}\hat{f}_{\alpha,\sigma}(x)\right| \leq B. \tag{3.8}$$

2. For a given $\varepsilon\in(0,1)$ and any $\tau\in(0,\varepsilon/(8B))$ uniformly on $\alpha\in[\underline{\alpha},\overline{\alpha}]$, $\sigma\in[\underline{\sigma},\overline{\sigma}]$

$$I_\tau(\alpha,\sigma) < \frac{\varepsilon}{8}. \tag{3.9}$$

Proof. We need in the following general fact on *standard* stable densities' derivatives of order n ([3], p.106): for $x\in R^+$

$$|\frac{d^n}{dx^n}s(x,\alpha,\beta)| \leq \frac{1}{\pi\alpha}\Gamma(\frac{n+1}{\alpha})\cdot(\cos(\frac{\pi}{2}K(\alpha)\cdot\beta))^{-\frac{n+1}{\alpha}}, \tag{3.10}$$

where

$$K(\alpha) = \alpha - 1 + \operatorname{sign}(1-\alpha). \tag{3.11}$$

Here α and β are *exponent* and *asymmetry* of the standard stable density. Let us apply (3.10)-(3.11) to cases: 1) $0<\alpha<1$, $\beta=1$, $n=1$, and 2) $1<\alpha<2$, $\beta=0$, $n=1$.

<u>In case</u> 1)

$$\left|\frac{d}{dx}\hat{f}_{\alpha,1}(x)\right| \leq \frac{1}{\pi\alpha}\Gamma(\frac{2}{\alpha})\cdot\frac{1}{(\cos(\pi\alpha/2))^{2/\alpha}}. \tag{3.12}$$

Since $\Gamma(x)$ is an increasing function and $\cos(\pi\alpha/2)$ decreases (remind that $\alpha \in (0,1)$), therefore

$$\Gamma(\frac{2}{\alpha}) \le \Gamma(\frac{2}{\underline{\alpha}}),\ (\cos\frac{\pi\alpha}{2})^{-2/\alpha} \le (\cos\frac{\pi\overline{\alpha}}{2})^{2/\,\overline{\alpha}}. \tag{3.13}$$

The inequalities (3.12) and (3.13) imply $|\frac{d}{dx}\hat{f}_{\alpha,1}(x)| \le \frac{1}{\pi\underline{\alpha}}\cdot\Gamma(\frac{2}{\underline{\alpha}})\cdot\frac{1}{(\cos(\pi\overline{\alpha}/2))^{2/\alpha}}$. So, $|\frac{d}{dx}\hat{f}_{\alpha,1}(x)| = \sigma^{-2\alpha}\cdot|\frac{d}{dx}\hat{f}_{\alpha,1}(y)| \le (\underline{\sigma})^{-2/\underline{\sigma}}\frac{1}{\pi\underline{\alpha}}\cdot\Gamma(\frac{2}{\underline{\alpha}})\cdot\frac{1}{(\cos(\pi\overline{\alpha}/2))^{2/\alpha}}$, where $y = \sigma^{-1/\alpha}\cdot x$, which implies (3.8).

In case 2) we have $|\frac{d}{dx}\hat{f}_{\alpha,1}(x)| \le \frac{2}{\pi\alpha}\cdot\Gamma(\frac{2}{\alpha}) \le \frac{2}{\pi\underline{\alpha}}\cdot\Gamma(\frac{2}{\underline{\alpha}})$. Thus, counting by the same way we conclude that (3.8) holds.

Now, by the *Mean Value Theorem*, from (3.7) with the help of (3.8) we come for both cases to the same type inequality $I_\tau(\alpha,\sigma) = |\frac{d}{dx}\hat{f}_{\alpha,\sigma}(x)|_{x=\theta\cdot\tau}\cdot\tau \le \tau\cdot B$, where $\theta = \theta_\tau \in (0,1)$. Thus, from the last inequality for $\tau \in (0, \varepsilon/(8B))$ we obtain (3.9).

4. Preliminary estimations III. For a given integer $N > 1$ consider the following series: In case $0 < \alpha < 1$

$$\begin{cases} \hat{f}_{\alpha,\sigma,N}(x) = \frac{1}{\pi}\sum_{n=1}^{N}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\frac{\sigma^n}{x^{n\alpha+1}}\sin(\pi n\alpha), \\ \tau \in (0,1),\ x \in [\tau, 1/\tau],\ \alpha \in [\underline{\alpha}, \overline{\alpha}],\ \sigma \in [\underline{\sigma}, \overline{\sigma}]; \end{cases} \tag{4.1}$$

In case $1 < \alpha < 2$

$$\begin{cases} \hat{f}_{\alpha,\sigma,N}(x) = \frac{2}{\pi}\sum_{n=1}^{N}(-1)^{n-1}\frac{\Gamma(\frac{2n-1}{\alpha}+1)}{(2n-1)!}\frac{x^{2n-2}}{\sigma^{(2n-1)/\alpha}}, \\ \tau \in (0,1),\ x \in [\tau, 1/\tau],\ \alpha \in [\underline{\alpha}, \overline{\alpha}],\ \sigma \in [\underline{\sigma}, \overline{\sigma}]. \end{cases} \tag{4.2}$$

(4.1) and (4.2) represent the partial sums of series expansions for densities $\hat{f}_{\alpha,\sigma}(x)$, $0 < \alpha < 1$, and $\hat{f}_{\alpha,\sigma}(x)$, $1 < \alpha < 2$, respectively, For $\tau \in (0,1)$, $N > 1$, $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ we have

$$\int_\tau^{1/\tau}(\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma,N}(x))dx =$$

$$= \begin{cases} \frac{1}{\pi}\int_\tau^{1/\tau}(\sum_{n>N}(-1)^{n-1}\cdot\frac{\Gamma(n\alpha+1)}{n!}\frac{\sigma^n}{x^{n\alpha+1}}\sin(\pi n\alpha))dx & if \quad 0 < \alpha < 1, \\ \frac{2}{\pi}\int_\tau^{1/\tau}(\sum_{n>N}(-1)^{n-1}\frac{\Gamma(\frac{2n-1}{\alpha}+1)}{(2n-1)!}\frac{x^{2n-2}}{\sigma^{(2n-1)/\alpha}})dx & if \quad 1 < \alpha < 2. \end{cases} \tag{4.3}$$

Denote ${}_{\alpha,\sigma,N}(\tau) = \int_\tau^{1/\tau}|(\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma,N}(x))|dx.$

Lemma 4. *1. The integrals at the right-hand-side of (4.3) exist.*
2. For given $\varepsilon \in (0,1)$, $\tau \in (0,1)$ there is an integer $N > 1$ (N doesn't depend on α and σ) such that

$$J_{\alpha,\sigma,N}(\tau) < \frac{\varepsilon}{8} \tag{4.4}$$

Proof. Case $0 < \alpha < 1$. From (4.3) we have

$$\begin{aligned} J_{\alpha,\sigma,N}(\tau) &\le \frac{1}{\pi}\int\limits_{\tau}^{1/\tau}\Big(\sum_{n>N}\frac{\Gamma(n\alpha+1)}{n!}\frac{\sigma^n}{x^{n\alpha+1}}\Big) \le \\ &\le \frac{1}{\pi}(-\tau+\frac{1}{\tau})\cdot\sum_{n>N}\frac{\Gamma(n\alpha+1)}{n!}\overline{\sigma}^n\cdot\tau^{-(n\overline{\alpha}+1)}. \end{aligned} \tag{4.5}$$

According to (4.5), with the help of (2.3) for N large enough

$$J_{\alpha,\sigma,N}(\tau) < \frac{2}{\pi\tau}(-\tau+\frac{1}{\tau})\overline{\alpha}^{1/2}\sum_{n>N}\left(\frac{c_r\tau(\overline{\alpha},\overline{\sigma})}{n^{1-\overline{\alpha}}}\right)^n, \tag{4.6}$$

where $c_\tau(\alpha,\sigma) = \exp(1-\alpha)\cdot\sigma\cdot\tau^{-\alpha}$. At the right-hand-side of (4.6) we have a convergent series, which proves the statement 1 of Lemma 4. The statement 2 is proved too because the last series doesn't depend on α and σ.

Case $1 < \alpha < 2$. From (4.3) we have

$$J_{\alpha,\sigma,N}(\tau) < \frac{2}{\pi\tau}(-\tau+\frac{1}{\tau})\overline{\alpha}^{1/2}\sum_{n>N}\left(\frac{c_r\tau(\overline{\alpha},\overline{\sigma})}{n^{1-\overline{\alpha}}}\right)^n, \tag{4.7}$$

A similar to case $0 < \alpha < 1$ estimations of series at the right-hand-side of (4.7) with the help of (2.3) imply the statements 1 and 2 of Lemma 4 in this case.

5. Solution to problem. In conditions of Theorem 1 choose for a given $\varepsilon \in (0,1)$ a number τ satisfying restrictions $\tau \in (0, \varepsilon/(8B))$, $\frac{1}{\tau} > x_0$. Then, for $|\alpha-\alpha'| + |\sigma-\sigma'|$ small enough from Lemmas 1 and 3 we have the following inequalities.

1.
$$\begin{aligned} &\int\limits_{0-}^{\tau}|\hat{f}_{\alpha,\sigma}(x)-\hat{f}_{\alpha',\sigma'}(x)|dx \le \int\limits_{0-}^{\tau}|\hat{f}_{\alpha,\sigma}(0)-\hat{f}_{\alpha',\sigma'}(0)|dx + \\ &+\int\limits_{0-}^{\tau}|\hat{f}_{\alpha,\sigma}(x)-\hat{f}_{\alpha,\sigma}(0)|dx + \int\limits_{0-}^{\tau}|\hat{f}_{\alpha',\sigma'}(x)-\hat{f}_{\alpha',\sigma'}(0)|dx = \\ &= I_\tau(\alpha,\sigma)+I_\tau(\alpha',\sigma')+\tau\cdot\gamma(\alpha,\sigma;\alpha'\sigma') < \frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8} = \frac{3\varepsilon}{8}, \end{aligned} \tag{5.1}$$

where the monotonity of $\hat{f}_{\alpha,\sigma}$ around the origin (point zero) and (3.1), (3.6) were used.

2.
$$\begin{aligned} &\int\limits_{1/\tau}^{+\infty}|\hat{f}_{\alpha,\sigma}(x)-\hat{f}_{\alpha',\sigma'}(x)|dx \le \int\limits_{1/\tau}^{+\infty}\hat{f}_{\alpha,\sigma}(x)dx + \int\limits_{1/\tau}^{+\infty}\hat{f}_{\alpha',\sigma'}(x)|dx = \\ &= (1-\hat{F}_{\alpha,\sigma}(\frac{1}{\tau})) + (1-\hat{F}_{\alpha',\sigma'}(\frac{1}{\tau})) < \frac{\varepsilon}{16}+\frac{\varepsilon}{16} = \frac{\varepsilon}{8}. \end{aligned} \tag{5.2}$$

In accordance with (5.1) and (5.2), for given $\varepsilon \in (0,1)$ and already chosen τ from (1.3) we obtain the following inequality (ε and τ are fixed)

$$E(\alpha,\sigma;\alpha',\sigma') = \int\limits_{0-}^{+\infty} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx =$$

$$= \int\limits_{0-}^{\tau} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx + \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx + \int\limits_{1/\tau}^{+\infty} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx \leq$$

$$\leq \frac{\varepsilon}{2} + \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx. \tag{5.3}$$

Now, let us choose an integer $N > 1$ such that for given ε and τ (4.4) takes place, and fix N. Then, by (5.3) and (4.4), for $|\alpha - \alpha'| + |\sigma - \sigma'|$ small enough we come to the following inequalities

$$\begin{aligned} E(\alpha,\sigma;\alpha',\sigma') &= \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)|dx + \frac{\varepsilon}{2} + \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)|dx + \\ &+ \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha',\sigma'}(x) - \hat{f}_{\alpha',\sigma',N}(x)|dx \leq \frac{3\varepsilon}{4} + \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)|dx. \end{aligned} \tag{5.4}$$

If we proceed as above in cases $E_1(\alpha;\sigma,\sigma')$ and $E_2(\sigma';\alpha,\alpha')$ then we obtain the following analogs of (5.4)

$$E_1(\alpha;\sigma,\sigma') \leq \frac{3\varepsilon}{4} + \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha,\sigma',N}(x)|dx \tag{5.5}$$

$$E_2(\sigma';\alpha,\alpha') \leq \frac{3\varepsilon}{4} + \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma,N}(x)|dx \tag{5.6}$$

In (5.6) we take σ' instead of, which changes nothing.

If we prove that for given ε, τ, N and $|\alpha-\alpha'|+|\sigma-\sigma'|$ small enough in conditions of Theorem 1

$$T_N^{(1)}(\tau) = \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha,\sigma',N}(x)|dx < \frac{\varepsilon}{4}, \tag{5.7}$$

$$T_N^{(2)}(\tau) = \int\limits_{\tau}^{1/\tau} |\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma,N}(x)|dx < \frac{\varepsilon}{4}, \tag{5.8}$$

then (5.5), (5.7) and (5.6), (5.8) imply (1.7) and (1.8), respectively, which due to Remark 1 proves Theorem 1.

Indeed, assuming that (5.7) and (5.8) take place, we may rewrite (5.5) and (5.6) in the forms

$$0 \le E_i(\alpha;\sigma,\sigma') < \varepsilon,\ \ i = 1,2, \tag{5.9}$$

respectively, for $|\alpha-\alpha'|+|\sigma-\sigma'|$ small enough. Tending $|\alpha-\alpha'|+|\sigma-\sigma'| \to 0$ from (5.9) we obtain $0 \le \overline{\lim}_{|\sigma-\sigma'|\to 0} \delta_i(\alpha;\sigma,\sigma') < \varepsilon$. Now, tending $\varepsilon \downarrow 0$ we prove Theorem 1.

Remark 2. Since in (4.1) and (4.2) under the signs of sums continuous functions on α and σ are written and (α,σ) belongs to compacts $B_1 = \{(\alpha,\sigma) : 0 < \underline{\alpha} \le \alpha \le \overline{\alpha} < 1, 0 < \underline{\sigma} \le \sigma \le \overline{\sigma} < +\infty\}$ and $B_2 = \{(\alpha,\sigma) : 1 < \underline{\alpha} \le \alpha \le \overline{\alpha} < 2, 0 < \underline{\sigma} \le \sigma \le \overline{\sigma} < +\infty\}$ respectively, then according to Cantor Theorem, they are uniformly continuous on these compacts. Therefore, $\hat{f}_{\alpha,\sigma,N}(x)$ for $0 < \alpha < 1$ and $1 < \alpha < 2$, as finite sums of uniformly continuous functions on B_1 and B_2 respectively, are also uniformly continuous on these compacts. Hence, for fixed τ the relations (5.7) and (5.8) take place.

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On Distance in Variation for Frequency Distributions Generated by Stable Laws

Two two-parametric families of densities generated by Stable Laws are considered. The stability by distance in variation by both parameters for the introduced families is established. With the help of these families their discrete analogs are constructed and the obtained result is proved for the analogs.

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О расстоянии по вариации для параметрических частотных распределений, порожденных устойчивыми плотностями

Рассмотрены два двухпараметрических семейства плотностей, порожденных устойчивыми законами. Установлена устойчивость в смысле расстояния по вариации по обоим параметрам для введенных семейств. С помощью этих семейств построены их дискретные аналоги, для которых доказан тот же результат.

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Կայուն խտություններով ծնված պարամետրական հաճախականային բաշխումների՝ ըստ վարիացիայի հեռավորության մասին

Դիտարկվել են կայուն խտություններով ծնված պարամետրական հաճախականային բաշխումների երկու ընտանիքներ: Ապացուցվել է դրանց, ըստ երկու պարամետրերի, կայունությունը՝ ըստ վարիացիայի հեռավորության: Կառուցվել են նաեւ այդ ընտանիքների դիսկրետ անալոգները, որոնց համար ապացուցվել է նույն արդյունքը:

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