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On Some Ill Conditioned Operator Equations

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1. Let $H, \langle \bullet, \bullet \rangle$ be a Hilbert space and A - a (bounded, linear) operator, acting in H. One of the most familiar ways of solving the equation

$$Ax = b \tag{1.1}$$

is the (Richardson's) iterative method

$$x_{n+1} = x_n - \alpha_n (Ax_n - b), \quad n \in \mathrm{Z}^+, \tag{1.2}$$

where x_0 is an initial guess and α_n is a numerical parameter. Denote by b_n the difference $Ax_n - b$ said to be the discrepancy or the residual. Then

$$b_{n+1} = (I - \alpha_n A) b_n = \prod_{k=0}^{n} (I - \alpha_k A) b_0. \tag{1.3}$$

For the simplest case $\alpha_n = \alpha = const$ (so called Richardson's stationary iterative process) and

$$b_n = (I - \alpha A)^n b_0.$$

If there exists a complex number α such that $||I - \alpha A|| < 1$, then the sequence $\{b_n\}$ tends to zero at least as a geometric progression and $\{x_n\}$ converges to the solution of (1.1).
Denote by

$$m(A) = m \stackrel{def}{=} \inf_{t \in \mathrm{C}} ||I - tA||$$

and

$$p(A) = p \stackrel{def}{=} \inf_{Ax \neq \theta} \frac{|\langle Ax, x \rangle|}{||Ax|| \cdot ||x||}.$$

The following results are from [1].

a) $m^2 + p^2 = 1$, so $m(A) < 1$ if and only if $p(A) > 0$.

b) Let $s = s(A)$ be the complex number such that $||I - s(A)A|| = m(A)$. If $\{z_n\}$ is a sequence of elements satisfying

$$\lim_{n\to\infty} \frac{|\langle Az_n, z_n\rangle|}{||Az_n|| \cdot ||z_n||} = p,$$

then

$$\lim_{n\to\infty} \frac{\langle z_n, Az_n\rangle}{||Az_n||^2} = s. \tag{1.4}$$

c) If $m(A) < 1$, then the operator A is invertible

$$m(A) = m(A^{-1}) \quad \text{and} \quad s(A) \cdot s(A^{-1}) = p^2$$

d) $k \le \frac{1+m}{1-m}$, where $k = k(A) \stackrel{def}{=} ||A|| \cdot ||A^{-1}||$ is the condition number of the operator A.

The set

$$W(A) = \left\{ \frac{\langle Ax, x\rangle}{||x||^2} : x \ne \theta \right\}$$

is the numerical range of the operator A and

$$W_n(A) = \left\{ \frac{\langle Ax, x\rangle}{||Ax|| \cdot ||x||} : Ax \ne \theta \right\}$$

is said (cf. [2]) to be the normalized numerical range of A. In [1] it is proved that the origin O belongs to the closure $\overline{W}(A)$ of the numerical range if and only if it belongs to $\overline{W}_n(A)$. So the condition $O \notin \overline{W}(A)$ is sufficient in order the iterations (1.2) converge to the solution a.

Formula (1.4) may be useful in theoretical speculations. In practice the parameter α is chosen to minimize the residual at each step, i.e.

$$\alpha_n = \frac{\langle b_n, Ab_n\rangle}{||Ab_n||^2}. \tag{1.5}$$

As

$$||b_{n+1}|| = ||b_n - \alpha_n Ab_n|| \le ||b_n - s(A) \cdot Ab_n|| \le m(A) \cdot ||b_n||,$$

then if $m(A) < 1$, the iterative process (1.2), (1.5) converges and

$$||x_n - a|| \le ||A^{-1}|| \cdot ||b_n|| \le ||A^{-1}|| \cdot m^n(A) \cdot ||Ax_0 - b|| \le \frac{k(A)}{||A||} \cdot m^n(A) \cdot ||Ax_0 - b||,$$

therefore

$$||x_n - a|| \le \frac{1 + m(A)}{1 - m(A)} \cdot \frac{||Ax_0 - b||}{r(A)} \cdot m^n(A), \tag{1.6}$$

where $r(A)$ is the spectral radius of A.

Let S be an invertible operator, $B = S^{-1}AS$, $c = S^{-1}b$. The solution to

$$By = c$$

(if $O \notin \overline{W}(B)$) may be sought as

$$y_{n+1} = y_n - \delta_n(By_n - c),$$

Multiplying the above relation from the left by S and denoting $Sy_n = x_n$ we get

$$x_{n+1} = x_n - \delta_n(Ax_n - b). \tag{1.7}$$

If the sequence $\{y_n\}$ converges, the sequence $\{x_n\}$ converges also.

It is known [3], Theorem 3, that if A is not a scalar multiple of the identity and $\mathcal{K}$ is a compact subset of the complex plane C then there is an invertible operator S such that $\text{int}W(S^{-1}AS) \supset \mathcal{K}$ (int here means the set of interior points). As the passage to a similar operator in fact means an equivalent renorming of the space, Theorem 3 means that the condition $O \notin \overline{W}(A)$ (equivalent to $m(A) < 1$) may be fulfilled by a pure chance.

In positive direction goes Theorem 2 from [3], stating that for any open convex set U containing the spectrum SpA of A there exists an invertible operator S with $\overline{W}(S^{-1}AS) \subset U$. This theorem implies that the intersection of closures of numerical ranges of all operators, similar to A coincides with the convex hull $chSpA$ of the spectrum of A, a result, proved by Hildebrandt in [4].

Localization of the spectrum of an operator, as compared with the normalized numerical range is easier to establish. For example, in finite dimensional space different Gershgorin type results may be used. In the general case the Bendixon-Hirsch theorem gives some pertinent information.

As we have seen above, if $O \notin \overline{W}(A) \backslash chSpA$ then iterations (1.7) may converge for an appropriate choice of δ_n. In next section we will consider this problem in more detailed way. In the last part a necessary and sufficient condition is found in order the condition number of the operator be equal to its upper bound.

2. As any compact convex subset F of C is the intersection of all closed circles, containing F, then condition $O \notin F$ implies that there exists at least one circle, containing F and separating it from the origin. It is easy to see that for any such circle the distance $|OC|$ from the origin to the centre of the circle is greater than its radius r, i.e. $\dfrac{r}{|OC|} < 1$.

Lemma 1. *Let F be a compact convex subset of* C *and $O \notin F$. Then there exists the unique closed circle $D = D(C, R)$, such that*

$$F \subset D \quad \textit{and} \quad O \notin D, \tag{2.1}$$

having the least ratio $\frac{R}{|OC|}$ *among all circles, satisfying condition (2.1).*
Further the circle described in the above Lemma will be said optimal.

Lemma 2. *The boundary of the optimal circle contains at least two points, belonging to F.*

Corollary. *For any circle F the optimal circle coincides with F.*

Example 1. *Let F be a segment $[\lambda_1, \lambda_2]$ in the complex plane* C, *which does not contain the origin O. Then the centre C of the optimal circle is at the point* $\frac{|\lambda_1| + |\lambda_2|}{sgn\lambda_1 + sgn\lambda_2} \exp(i(\arg\lambda_1 + \arg\lambda_2))$ *and the ratio $R/|OC|$ is equal to* $\frac{|\lambda_1 - \lambda_2|}{|\lambda_1| + |\lambda_2|}$.

Example 2. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.5+i & 0 & 0 \\ 0 & 0 & 1.5+i & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The optimal circle, containing SpA is centered at the point $C(\frac{13}{6}; 0)$ and has the radius $R = \frac{\sqrt{13}}{3}$, so $m(A) = \frac{2}{\sqrt{13}} = 0.5547\cdots$. For this operator $k(A) = 2$ and $\frac{1+m}{1-m} = 3.4914\cdots$.

As we have seen above, if $O \notin chSpA$, then for $\alpha = \frac{1}{z_0}$, where z_0 is the affix of the optimal circle centre, the spectrum of the operator $B = I - \alpha A$ lies in a circle, centered at the origin and of radius, equal to $\frac{R}{|z_0|}$, which is strictly less than 1.

Proposition. *Let B be an operator, acting in a Hilbert space $H, \langle \bullet, \bullet \rangle$ and ε - a positive number. Then there exists a scalar product $[\bullet, \bullet]$, generating a new norm $|\bullet|$ equivalent to the former one and $|B| \leq r(B) + \varepsilon$.*

Remark. This Proposition is a slight modification of § 1, 1.4 from [5].

Summarizing, we propose the following way of solving equation (1.1) in the case when iterative process (1.2), (1.5) fails to converge or converges slowly.

1. Localize the spectrum of A,
2. find the optimal circle $D(z_0, R)$,
3. choose a positive number and introduce an equivalent norm such that

$$\left\| I - \frac{1}{z_0} A \right\|_{new} \leq \frac{R}{|z_0|} + \varepsilon < 1,$$

4. do iterations until the desired precision is achieved.

As it has been noted above, the above program may be realized, if $O \notin chSpA$. This limitation is due to the fact that we consider only the first order polynomials $P_1(z) = 1 - \alpha z$ to minimize the residual. Denote by $\sigma(A)$ the full spectrum of A, i.e. the union of SpA and all bounded connectivity components of C $\setminus SpA$. As it is well known (cf. [6], Ch. III, Lemma 1.3) the set $\sigma(A)$ is polynomially convex, i.e. for any point $z \in \mathrm{C} \setminus \sigma(A)$ there exists a polynomial P, such that

$$|P(z)| > \sup_{\xi \in \sigma(A)} |P(\xi)|.$$

If $O \notin \sigma(A)$, then there exists a polynomial Q such that $Q(0) = 1$ and

$$\sup_{\xi \in SpA} |Q(\xi)| = \sup_{\xi \in \sigma(A)} |Q(\xi)| < 1.$$

By appropriate choice of the polynomial the last supremum may be set arbitrary small. Consider the function $h(z) = \dfrac{1}{z}$. Evidently, h is analytic everywhere, excluding the origin, so it may be approximated with any precision on $\sigma(A)$ by polynomials, i.e. $\sup_{z \in \sigma(A)} |P(z) - h(z)| \leq \varepsilon$. Then $\sup_{z \in \sigma(A)} |zP(z) - 1| \leq \varepsilon \cdot \sup_{z \in \sigma(A)} |z| = \varepsilon \cdot r(A)$, so for any positive δ there exists a polynomial $Q(z) = 1 - zP(z)$, such that $Q(0) = 1$ and $\sup_{z \in \sigma(A)} |Q(z)| \leq \delta$.

Despite the less restrictive condition imposed on SpA, the use of higher order polynomials (and instead of optimal circles the search of Cassini's ovals) is hindered by the lack of relevant theory. Nevertheless, the truncated Faber series gives almost the best result (cf. [7], 18.2).

Example 3. *Let A be the operator, defined by the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Standard calculations show that

$$||tA - I|| = |t| + \sqrt{|1-t|^2 + |t|^2}.$$

As

$$|t| + \sqrt{|1-t|^2 + |t|^2} \geq |t| + |1-t| \geq 1$$

then $m(A) = 1$. For the equation

$$Ax = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

ordinary iterative method for an initial guess $x_0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ shows completely chaotic character.

As $SpA = \{1\}$, then $B = A - I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. We have $B^2 = 0$ and

$$[f, g] = \langle f, g \rangle + \varepsilon^{-2} \langle Bf, Bg \rangle = \langle f, g \rangle + 4\varepsilon^{-2} f_2 \overline{g}_2.$$

The modified algorithm gives (in the finite-precision arithmetic) the exact solution in a few (5-6, depending on ε) steps.

3. In [8] it is shown that for any two Hilbert space operators A and B the equality $||A+B|| = ||A|| + ||B||$ holds if and only if $||A|| \cdot ||B|| \in \overline{W}(B^*A)$. So

$$|s(A)| \cdot ||A|| = ||s(A)A - I + I|| = ||s(A)A - I|| + 1 = m(A) + 1$$

and

$$|s(A^{-1})| \cdot ||A^{-1}|| = ||s(A^{-1})A^{-1} - I + I|| = ||s(A^{-1})A^{-1} - I|| + 1 = m(A) + 1,$$

implying

$$k(A) = \frac{1 + m(A)}{1 - m(A)}$$

if and only if $m(A) \in \overline{W}(s(A)A - I)$ and $m(A) \in \overline{W}(s(A^{-1})A^{-1} - I)$.

If $m(A) \in W(s(A)A - I)$ and $m(A) \in W(s(A^{-1})A^{-1} - I)$ (these conditions follow from above inclusions in any finite dimensional space, as the numerical range in this case is closed) one gets a more transparent picture.

In [5] (Corollary 2.1) the following assertion is proved.

Let $\lambda \in W(A)$ *and* $|\lambda| = ||A||$. *Then there exists an element* x *such that* $Ax = \lambda x$, $A^*x = \lambda x$. Recall that in this case λ is said to be a reducing (or normal) eigenvalue of the operator A.

As $||s(A)A - I|| = m(A)$ and $||s(A^{-1})A^{-1} - I|| = m(A)$, then $s(A)Ax = (1 + m(A))x$ and $s(A^{-1})A^{-1}y = (1 + m(A))y$, meaning that $\dfrac{1 + m(A)}{s(A)}$ and $\dfrac{s(A^{-1})}{1 + m(A)} = \dfrac{1 - m(A)}{s(A)}$ are reducing eigenvalues of A. Therefore

$$\begin{pmatrix} \dfrac{1+m(A)}{s(A)} & 0 \\ 0 & \dfrac{1-m(A)}{s(A)} \end{pmatrix} \oplus B$$

Note that $\dfrac{1 + m(A)}{s(A)} = ||A||$ and $\dfrac{1 - m(A)}{s(A)} = ||A^{-1}||^{-1}$.

The general form of such operator is

$$A = e^{i\alpha}\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \oplus B\right),$$

where $\alpha \in$ R, $\lambda_1 > \lambda_2 > 0$ and $\left|\left|\dfrac{2}{\lambda_1 + \lambda_2} B - I\right|\right| \leq \dfrac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$.

Example 3. *Let*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.5 + 0.5i & 0 & 0 \\ 0 & 0 & 1.5 - 0.5i & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Then $m(A) = \dfrac{1}{3}$, $s(A) = \dfrac{2}{3}$, $s(A^{-1}) = \dfrac{4}{3}$, $k(A) = 2$.

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Լ. Զ. Գևորգյան

Վատ պայմանավորված օպերատորային հավասարումների մասին

Որոշ դեպքերում, երբ օպերատորային հավասարման լուծման սովորական իտերացիոն եղանակը դանդաղ է զուգամիտում կամ չի զուգամիտում, առաջարկվում է տարածության համարժեք վերանորմավորում, ինչը հանգեցնում է վերահսկելի արագությամբ զուգամիտության: Գտնված է նաև անհրաժեշտ և բավարար պայման, որպեսզի օպերատորի պայմանավորվածության թիվը և նվազագույն նորմը կապող անհավասարությունում տեղի ունենա հավասարություն:

Л. З. Геворгян

О некоторых плохо обусловленных операторных уравнениях

В некоторых случаях, когда обычный итерационный процесс решения операторных уравнений медленно сходится или расходится, предлагается эквивалентная перенормировка пространства, ведущая к сходимости с контролируемой скоростью. Найдено также необходимое и достаточное условие, при котором в неравенстве между числом обусловленности и минимальной номой достигается равенство.

Литература

1. *L. Gevorgyan* — Math. Sci. Res. J. 2004. Vol. 8. P. 16-26.
2. *W. Auzinger* — Applied Numerical Mathematics. 2003. Vol. 45. P. 367-388.
3. *J. P. Williams* — J. Math. Anal. Appl. 1969. Vol. 36. P. 307-314.
4. *S. Hildebrandt* — Math. Ann. 1966. Vol. 163. P. 415-421.
5. *M. A. Krasnoselsky et al.* Approximate solutions of operator equations, Wolters-Nordhoff, Gröningen. 1972.
6. *T. Gamelin* Uniform Algebras, Prentice-Hall, Englewood Cliffs. N. J. 1969.
7. *P. Henrici* — Applied and Computational Complex Analysis. V. 3. Wiley-Interscience. 1986.
8. *M. Barraa, M. Boumazgour* — Proc. Amer. Math. Soc. 2002. V. 130. N 2. P. 471-476.
9. *L. Gevorgyan* — Electronic Journal "Investigated in Russia". 2003. V. 20. P. 191-200. http://zhurnal.ape.relarn.ru/articles/2003/020.pdf