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MATHEMATICS

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Frequency Distributions in Growing Biomolecular Networks Based on Stable Densities

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1. The Scale-Invariant and Semi-Group Properties. One of the important regularities of many large-scale biomolecular systems is their *itself organization*. The conception of the self-organization appears in the Phase Transition System Theory where, very often, systems spontaneously self-organize themselves in fractals [1]. The similar situation may be observed in networks, in particular, biomolecular networks. Here we often see the reproduction of properties of the networks on previous fractal during the process of network's enlargement.

Let us more precisely introduce *self-organized* systems. In *self-organized* systems knowing the *local* frequency distribution in two successive non-intersected intervals (fractals) we must able to obtain the frequency distribution in the *united interval* (in the union of these intervals). It implies that we can *extrapolate* the frequency distribution in *whole* system.

Definition 1. *We say that a random variable ξ of events exhibits the* ***Power Law*** $\{p_n\}$ *if*

$$p_n = P\{\xi = n\} = c(\rho) \cdot n^{-\rho},\ 1 \leq \rho < +\infty,\ n = 1, 2, \dots, \tag{1.1}$$

where the normalization factor $c(\rho)$ *takes the value*

$$c(\rho) = \left(\sum_{n \geq 1} n^{-\rho} \right)^{-1} \tag{1.2}$$

and P denotes probability.

A Power Law (1.1)-(1.2) was found to describe various events in the vicinity of critical points in physical and chemical *phase transition systems*. For such systems Power Law is of interest also because of its *genetic* property in the case of self-organization, which can be explained as follows. The selection process cannot avoid the order exhibited by most members of the system. Here it is of importance that Power Law is *scale-invariant*, implying that the knowledge of statistical properties of any part of the complex system allows to extrapolate these properties to whole system.

The *scale-invariant property* means that the replacement of variable n in p_n by a new variable $m = s \cdot n$ with arbitrary positive integer s doesn't change the functional form of frequency distribution. For Power Law we have

$$p_m = \frac{1}{c(\rho)} \cdot p_s \cdot p_n \text{ (see (1.1)).}$$

The *second* regularity in self-organized systems is of the following type. The frequency distribution must be of the *same* type in united interval as it is in each interval. These intervals (fractals) may be chosen with *approximately equal lengths* in the way, which allows to postulate either the *independence* or some type of *"weak" dependence* between the numbers of events' occurrences on each fractal. These random numbers are characterized by local frequency distributions on fractals. In *self-organized* systems of such type for densities of continuous analogs of events' occurrence numbers' distributions, instead of *scale-invariance* the *semi-group property* has to take place. In contradiction to *scale-invariant* property, where the operation of *multiplication* is used, the *semi-group* property is based on operation of *convolution*.

So, we formulate arguments, which allow to replace in many cases the *scale-invariant* by *semi-group* property for growing biomolecular networks.

The *semi-group* property implies that the convolution of densities (or distribution functions) of the *same* type equals to density (or distribution functions) of exactly *this* type. Such *semi-group* property *intrinsic* for normal, Cauchy's, Levy's distribution functions and for many other very useful ones.

Notice that the *semi-group* property holds, for instance, for very important in Probability Theory *four-parametric* family of *Stable Laws* [2, 3]. That is the reason why we are going to use some *stable laws* as continuous analogs for frequency distributions arising in growing biomolecular networks. Such probability distributions, in particular, satisfy the semi-group property.

Later we are going to substantiate that for *self-organized* large-scale biomolecular systems, in particular, for growing biomolecular networks of above described type the conception of *regular variation* and *semi-group* property for empirical frequency distributions' continuous analogs are *closely connected and supplement each other* from the point of view of Probability Theory.

It is just the time to mention that *Stable Laws* not only satisfy the *semi-group* property, but also have regularities which are characteristic for empirical frequency distributions in large-scale biomolecular systems. Here they are: *1) Regular Variation; 2) skewness; 3) satisfaction to some convexity properties; 3) Power Law behavior; 5) unimodality.* [4]

In order to formulate the *semi-group* property for distribution functions *mathematically* we use the concept of *convolution* (see, for instance [3]).

Let $\{F_\alpha(x)\}$ be a *one-parametric* family of distributions and f_α be the density of F_α. So, we have *one-parametric* family of densities $\{f_\alpha(x)\}$.

Definition 2. *We say that for a family $\{f_\alpha(x)\}$ the **semi-group** property holds if this family is **closed** under convolution, i.e. $f_{\alpha_1} * f_{\alpha_2} = f_{\alpha_1+\alpha_2}$, where "$*$" denotes the sign of convolution.*

2. Stable Laws. In this *Section* we introduce another, *more powerful* than the *semi-group* property and, obviously, more *restrictable* property, which extracts the well-known in Probability Theory family of *Stable Laws.* [3]

Definition 3. *We say that the distribution function S is **stable** if for any, $a_1 \in \mathrm{R}^1$, $a_2 \in \mathrm{R}^1$, $b_1 \in \mathrm{R}^+$, $b_2 \in \mathrm{R}^+$, there are numbers $a \in \mathrm{R}^1$, $b \in \mathrm{R}^+$ such that*

$$S\left(\frac{x-a_1}{b_1}\right) * S\left(\frac{x-a_2}{b_2}\right) = S\left(\frac{x-a}{b}\right),\ x \in \mathrm{R}^1. \tag{2.1}$$

Let us consider *gamma*-distribution functions

$$F_{\alpha,\nu}(x) = \begin{cases} \dfrac{1}{\Gamma(\nu)}\alpha^\nu \cdot \displaystyle\int_0^x u^{v-1}e^{-\alpha u}du, & x \in \mathrm{R}^1, \\ 0, & x \leq 0. \end{cases}$$

$F_{\alpha,\nu}(x)$ may be represented as $F_{\alpha,\nu}(x) = \dfrac{\gamma(\nu,\alpha x)}{\Gamma(\nu)}$, where $\Gamma(\cdot)$ is the Euler's *Gamma Function* and $\gamma(\nu,x) = \int_0^x e^{-t} \cdot t^{\nu-1}dt$ - a well known in Mathematical Analysis *incomplete Gamma Function*. Easily seen that the Gamma-densities satisfy the *semi-group property*, but the Gamma distribution function is not *stable* [3]. Indeed, taking the density

$$f_{\alpha,\nu}(x) = \frac{1}{\Gamma(\nu)}\alpha^\nu \cdot x^{\nu-1}e^{-\alpha x},\ \nu \in \mathrm{R}^+,\ x \in \mathrm{R}^+ \tag{2.2}$$

with $\alpha = 1$ and ν is fixed let us check out (2.1) for $a_1 = a_2 = 0$, $b_1 = b_2 = 1$. Then (2.1), due to (2.2), in our case in terms of densities takes the form

$$f_{1,2\nu}(x) = f_{1,\nu}(x) * f_{1,\nu}(x) = \frac{1}{b}f_{1,\nu}\left(\frac{x-a}{b}\right),\ x \in \mathrm{R}^+$$

with some admissible constants $a \in \mathrm{R}$, $b \in \mathrm{R}^+$. Note that here also the *semi-group property* for *Gamma* densities was used. But the last equality, i.e. $f_{1,2\nu}(x) = \frac{1}{b} f_{1,\nu}\left(\frac{x-a}{b}\right)$ for any a and b cannot be true for all values $x \in \mathrm{R}^+$.

It is known (see, for instance, [3]) that the *standard Normal*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \; x \in \mathrm{R}^1,$$

Levy's

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \frac{1}{\sqrt{u^3}} e^{-1/2u} du, \; x \in \mathrm{R}^+,$$

Cauchy's

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x, \; x \in \mathrm{R}^1,$$

distribution functions are *stable.*

Historically the family of *Stable Laws* has been extracted from the class of *Infinitely Divisible Distributions*. More precisely, the *Canonical Representations* for the logarithm of characteristic function of *Infinitely Divisible Distributions* has been found. It allowed to obtain the logarithm of characteristic function for all *Stable Laws*. Such a Canonical Representation of Stable Laws became a *powerful tool* for many new results. For instance, the existence of *continuous densities* for *Series Expansions* for them.

Unfortunately the *Normal, Levy's, Cauchy's* distributions are the only ones, which can be represented in closed form. For others there are *Series Expansions* and *Integral Representations* [2,3].

Let us characterize the parameters of the family of *Stable Laws.*

Definition 4. *We say that the distribution functions F_1 and F_2 belong to the* ***same class*** *if there are constants $a \in \mathrm{R}^1$ and $b \in \mathrm{R}^+$ such that for all $x \in \mathrm{R}^1$ the equality holds*

$$F_1(x) = F_2\left(\frac{x-a}{b}\right).$$

If $S(x)$ is a Stable Law, then $S\left(\frac{x-a}{\sigma}\right)$ with $a \in \mathrm{R}^1$ and $\sigma \in \mathrm{R}^+$ is also *Stable*. Therefore, we faced with first two *parameters* of Stable Laws: the *shifting parameter* (a) and the *scale factor* (σ). More essential are two other parameters: the *exponent* and the *asymmetry*. In order to define them, it is necessary to give another *equivalent* definition of Stable Laws [3]. Let us introduce the notation $\xi \stackrel{d}{=} \eta$ to indicate that random variables ξ and η are *identically distributed*. By this notation, $\eta \stackrel{d}{=} \alpha\xi + \beta$ means that distribution functions of ξ and η belong to the *same class*, and differ only by *location* parameters.

Let $\{\xi_n\}$ and ξ are independent identically distributed random variables and

$$S_n = \xi_1 + \xi_2 + \ldots + \xi_n, \; n \geq 1.$$

Now the following definition is equivalent to *Definition 3.*

Definition 5. *We say that* $S(x) = P(\xi < x)$ *is stable if for each integer* $n \geq 1$ *there are constants* $\sigma_n \in \mathrm{R}^+$ *and* $a_n \in \mathrm{R}^1$ *such that*

$$S_n \stackrel{d}{=} \sigma_n \xi + a_n. \tag{2.3}$$

It is amazing that in (2.3) necessarily (see [3])

$$\sigma_n = n^{1/\alpha};\ n \geq 1,\ \alpha \in (0, 2]. \tag{2.4}$$

The number α is called the *exponent* of Stable Law $S = S_\alpha$.

The values $\alpha = 2$, $\alpha = 1/2$, $\alpha = 1$,characterize in (2.3) *Normal, Levy's and Cauchy's Laws*, respectively, inside the family of Stable Laws.

So, now, we are familiar with *three parameters*. These three parameters being fixed, the forth one characterizes the *skewness* of a Stable Law.

Definition 6. *For a Stable Law* S *the limit*

$$\lim_{x\to+\infty} \frac{1 - S(x) - S(-x)}{1 - S(x) + S(-x)} \stackrel{def}{=} \beta \in [-1, 1]$$

is called ***asymmetry****.*

It is the second essential *parameter of Stable Laws*.

If now we take for each value of exponent $\alpha \in (0, 2)$ and asymmetry $\beta \in [-1, 1]$ *one representative* $S_{\alpha,\beta}$ from the family of *Stable Laws*, then the family may be represented in the form $\bigcup\limits_{(\alpha,\beta)} \{S_{\alpha,\beta}(\frac{x-a}{\sigma}) : s \in \mathrm{R}^+, a \in \mathrm{R}^1\}$.

For our further purpose we need either *Stable Laws* concentrated on $[0, +\infty)$, or symmetric *Stable Laws* which allow to construct distribution functions concentrated on $[0, +\infty)$. It means that we must take either $\beta = 1$ or $\beta = 0$.

3. Two Families of Densities. The search of distributions being applicable to large-scale biomolecular systems for the approximation of frequency distributions arising there are still continued [2,4].

Here, as continuous analogues of such distributions we suggest *two-parametric* families of densities, connected with *Stable Laws*, that possess known statistical facts 1-5 (see *Section 1*) on empirical frequency distributions in biomolecular systems.

Powerful tools for investigation of asymptotic properties of *Stable Laws* are their *Integral Representations*. Let us give a brief information on this topic.

Family 1. Let

$$\{\hat{f}_{\alpha,\sigma}(x) = \sigma^{-1/\alpha} \cdot s(x \cdot \sigma^{-1/\alpha}; \alpha; 1) :\ 0 < \alpha < 1,\ \sigma \in \mathrm{R}^+\} \tag{3.1}$$

be a *two-parametric* family of *standard* stable densities concentrated on R^+ (see V. Feller [3] and V. Zolotarev [5]). Here $s(x; \alpha, 1)$ is given by the series expansion

$$s(x; \alpha, 1) = \frac{1}{\pi} \sum_{n\geq 1} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} \cdot \frac{1}{x^{n\alpha+1}} \cdot \sin(\pi n\alpha), \tag{3.2}$$

and α is it's *exponent.*

Family 2. Let

$$\{\hat{f}_{\alpha,\sigma}(x) = 2\sigma^{-1/\alpha} \cdot s(x \cdot \sigma^{-1/\alpha}; \alpha; 0) : \; 1 < \alpha \le 2, \; \sigma \in \mathrm{R}^+\} \tag{3.3}$$

be a *two-parametric* family of densities concentrated on R^+ and generated by standard symmetric *stable densities* $s(x;\alpha,0)$, namely

$$\begin{aligned} s(x;\alpha,0) &= \frac{1}{\pi}\sum_{n\ge1}(-1)^{n-1}\frac{\Gamma(\frac{n}{\alpha}+1)}{n!}\cdot\frac{1}{x^{n\alpha+1}}\cdot\sin(\frac{\pi n}{2}) = \\ &= \frac{1}{\pi}\sum_{m\ge1}(-1)^{2m}\frac{\Gamma(\frac{2m-1}{\alpha}+1)}{(2m-1)!}\cdot x^{2m-2}\cdot\sin\pi(m-\frac{1}{2}) = \\ &= \frac{1}{\pi}\sum_{m\ge1}(-1)^{m-1}\frac{\Gamma(\frac{2m-1}{\alpha}+1)}{(2m-1)!}\cdot x^{2m-2}. \end{aligned} \tag{3.4}$$

Here α is also is the *exponent* of these stable densities.

From general case, being considered in 2.7, p. 173, [2] for the families (3.1) and (3.3) we extract the following conclusions.

Theorem. *(a) The graphs of $\hat{f}_{\alpha,\sigma}(x)$ are* ***downward/upward convex****, and are* ***unimodal*** *with only one mode, say m, where $0 < m < \infty$ for $0 < \alpha < 1$ and $m = 0$ for $1 < \alpha \le 2$.*

(b) $\hat{f}_{\alpha,\sigma}(x) \approx \mathrm{const}\cdot\frac{1}{x^{\alpha+1}}$, $x \to \infty$.

It means that $\hat{f}_{\alpha,\sigma}(x)$ ***varies regularly*** *at infinity and exhibits constant slowly varying component. The exponent of regular variation, say $(-\rho)$, of $\hat{f}_{\alpha,\sigma}(x)$ and the* ***exponent*** *α of stable density which generates $\hat{f}_{\alpha,\sigma}(x)$ satisfy the equality $\rho = \alpha + 1$.*

Often it is more preferable to have *Integral Representations* instead of *Expansions*. From general expansions of *Integral Representations* [3,5], in our case, we obtain the following representations in case $\sigma = 1$.

Let $0 < \alpha < 1$ and $S(x;\alpha,1)$ be a distribution function of $s(x;\alpha,1) = \hat{f}_{\alpha,\sigma}(x)$. Then, for $x \in \mathrm{R}^+$ the representation

$$S(x;\alpha,1) = \frac{1}{2}\int_{-1}^{1}\exp(-\frac{1}{x^{\alpha(\alpha-1)}}\cdot U_\alpha(y))dy, \tag{3.5}$$

where

$$U_\alpha(y) = \left(\frac{\sin(\frac{\pi}{2}\alpha\cdot(y+1))}{\cos(\frac{\pi}{2}y)}\right)^{\frac{\alpha}{\alpha-1}} \cdot \frac{\cos(\frac{\pi}{2}((\alpha-1)\cdot y+\alpha))}{\cos(\frac{\pi}{2}y)}, \; y \in [-1,1] \tag{3.6}$$

holds. Now, making the variable's replacement $\varphi = \frac{\pi}{2}y$ in integral (3.5) and using equalities in (3.6)

$$\cos((\alpha-1)\varphi + \frac{\pi}{2}\alpha) = \cos(\frac{\pi}{2} - (1-\alpha)(\varphi+\frac{\pi}{2})) = \sin((1-\alpha)(\varphi+\frac{\pi}{2})),$$

$$\cos\varphi = \sin(\varphi + \frac{\pi}{2}),$$

we come to another representation for $S(x;\alpha,1)$. New variable's replacement $\psi = \varphi + \frac{\pi}{2}$ leads to the following *Integral Representation*

$$S(x;\alpha,1) = \frac{1}{\pi}\int_0^{\pi} \exp(-\frac{1}{x^{\alpha(\alpha-1)}} \cdot \overline{U}_\alpha(\psi))d\psi, \tag{3.7}$$

where

$$\overline{U}_\alpha(\psi) = \left(\frac{\sin(\alpha\psi)}{\sin\psi}\right)^{\frac{\alpha}{\alpha-1}} \cdot \frac{\sin((1-\alpha)\cdot\psi)}{\sin(\psi)},\ \psi \in [0,\pi]. \tag{3.8}$$

Now, let $1 < \alpha \leq 2$ and $S(x;\alpha,0)$ be a distribution function of $s(x;\alpha,0)$. Then, for $x \in \mathrm{R}^+$ we have

$$S(x;\alpha,0) = 1 - \frac{1}{2}\int_0^{1} \exp(-x^{\alpha(\alpha-1)} \cdot V_\alpha(y))dy, \tag{3.9}$$

where

$$V_\alpha(y) = \left(\frac{\cos(\frac{\pi}{2}y)}{\sin(\frac{\pi}{2}\alpha y)}\right)^{\frac{\alpha}{\alpha-1}} \cdot \frac{\cos(\frac{\pi}{2}(\alpha-1)\cdot y)}{\cos(\frac{\pi}{2}y)},\ y \in [0,1]. \tag{3.10}$$

Now, making the variable's replacement $\psi = \frac{\pi}{2}y$ in integral (3.9) we come to the following *Integral Representation*

$$S(x;\alpha,0) = 1 - \frac{1}{\pi}\int_0^{\pi/2} \exp(-x^{\alpha(\alpha-1)} \cdot \overline{V}_\alpha(\psi))d\psi, \tag{3.11}$$

where

$$\overline{V}_\alpha(\psi) = \left(\frac{\cos(\psi)}{\sin(\alpha\psi)}\right)^{\frac{\alpha}{\alpha-1}} \cdot \frac{\cos((\alpha-1)\cdot\psi)}{\cos(\psi)},\ \psi \in [0,\pi/2]. \tag{3.12}$$

Next, from (3.7)-(3.8) and (3.11)-(3.12) we find the distribution function $\hat{F}_{\alpha,1}(x)$ for densities $\hat{f}_{\alpha,1}(x)$ being equal to $S(x;\alpha,1)$ in case $0 < \alpha < 1$ and to $2\cdot S(x;\alpha,0) - 1$ in case $1 < \alpha \leq 2$ on R^+, which is concentrated on $[0,+\infty)$.

Let us make one *remark*. Integral Representations for densities and other derivatives of *Stable Laws* are easy to derive by differentiating corresponding *Integral Representations of Stable Laws under the sign of integral*.

4. **Discretization.** The families (3.1) and (3.3) of densities are suggested as *continuous analogues* of desired frequency distributions for application to large-scale biomolecular networks.

The desired distributions are possible to derive with the help of the *discretization* procedure. Let us describe this procedure.

Let $f(x)$ be a distribution density concentrated on R^1. Then, the corresponding *discretization* (frequency distribution), say $\{p_n\}$ taking values only in points $\dots, -2, -1, 0, 1, 2, \dots$ has the following form

$$p_n = \begin{cases} \int_{n-1}^{n} f(x)dx, & n = 1, 2, \dots, \\ 0, & n = 0, \\ \int_{n}^{n+1} f(x)dx, & n = -1, -2, \dots . \end{cases} \tag{4.1}$$

or

$$p_n = \begin{cases} \int_{n}^{n+1} f(x)dx, & n = 1, 2, \dots, \\ \int_{-1}^{1} f(x)dx, & n = 0, \\ \int_{n-1}^{n} f(x)dx, & n = -1, -2, \dots . \end{cases} \tag{4.2}$$

In particular if $f(x)$ is *symmetric*, then $p_n = p_{-n}$, i.e. $\{p_n\}$ is also *symmetric*. The result of discretization (4.1) corresponding to the families (3.1) and (3.3) of densities extract the desired frequency distributions families.

We call $\{p_n\}$ *a discretization* of $f(x)$.

We are interested in constructing of frequency distributions from densities that either are concentrated on $[0, +\infty)$, or are symmetric. In the last case we transform the "mass" of density in $(-\infty, 0]$ into $[0, +\infty)$ and add it to the "mass" concentrated in $[0, +\infty)$.

According to series expansion (3.2) and (3.4) we obtain

$$p_n(\alpha, 1) = \frac{1}{\pi} \sum_{m \geq 1} (-1)^{m-1} \frac{\Gamma(m\alpha + 1)}{m!} \sin(\pi n \alpha) \cdot \int_{n-1}^{n} \frac{dx}{x^{m\alpha+1}} =$$

$$= \frac{1}{\pi} \sum_{m \geq 1} (-1)^{m} \frac{\Gamma(m\alpha + 1)}{m\alpha \cdot m!} (n^{-m\alpha} - (n-1)^{-m\alpha}) \sin(\pi n \alpha),$$

$$n = 0, 1, 2, \dots, \ 0 < \alpha < 1,$$

for the family (3.1), and

$$p_n(\alpha, 0) = \frac{1}{\pi} \sum_{m \geq 1} (-1)^{m-1} \frac{\Gamma(\frac{2m-1}{\alpha} + 1)}{(2m-1) \cdot (2m-1)!} (n^{2m-1} - (n-1)^{2m-1}), \ n = 1, 2, \dots,$$

$$p_0(\alpha, 0) = 0, \ 1 < \alpha < 2$$

for the family (3.3). Here the first form of discretization in (4.1) was used.

Now, it is necessary to check out the validity of known statistical fact for empirical frequency distributions.

It is obvious that after *discretization* procedure the properties of *unimodality* and *convexity* for $\{\{p_n(\alpha, 1)\} : 0 < \alpha < 1\}$ and $\{\{p_n(\alpha, 0)\} : 1 < \alpha < 2\}$ are conserved.

It is easy to understand by considering the *graphical* approach (see, in general, figure 1).

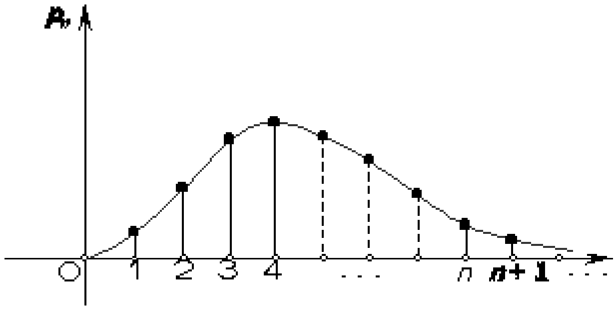

Figure 1.

In particular, the considered densities being decreasing at infinity imply the same property for *discretization*.

The fact of regular variation of *discretization* may be proved even in general case.

Given a density $f(x)$ concentrated on $[0, +\infty]$ decreasing at infinity and *varies regularly* at infinity with *exponent* α.

Definition 7. *Measurable function* $R(t) > 0$ *defined on* $(0, +\infty)$ *varies regularly at infinity with exponent* $\rho \in (-\infty, +\infty)$ *if for any* $x > 0$ *the limit exists*

$$\lim_{t \to +\infty} \frac{R(xt)}{R(t)} = x^{\rho} \tag{4.3}$$

If $R(t)$ varies regularly at infinity with exponent $\rho = 0$, then we call it *slow varying* at infinity and denote by $L(t)$.

Any regularly varying at infinity function $R(t)$ with exponent ρ is presented in the form

$$R(t) = t^{\rho} L(t), \; 0 < t < +\infty. \tag{4.4}$$

Any function of type (4.4) with $\rho \in (-\infty, +\infty)$ and with slowly varying $L(t)$, varies regularly at infinity with exponent ρ.

For a sequence of positive numbers $\{c_n\}$ the definition of regularly varying function with exponent ρ stays unchanged if in (4.3) we replace x and t by integers $s > 1$ and $n \geq 1$ respectively, i.e.

$$\lim_{n \to +\infty} \frac{c_{sn}}{c_n} = s^{\rho} \text{ (see, for instance, [6]).}$$

Assume that $\{p_n\}$ is a *discretization* of $f(x)$ of the form (4.1), i.e.

$$p_n = \int_{n-1}^{n} f(x)dx, \quad n = 0, 1, 2, \dots .$$

Then $\{p_n\}$ *varies regularly* at infinity with exponent α. Indeed, for integer $m > 1$ we have to prove the existence of limit

$$\lim_{n\to+\infty}\frac{p_{nm}}{p_n}=\lim_{n\to+\infty}\frac{\displaystyle\int_{nm-1}^{nm}f(x)dx}{\displaystyle\int_{n-1}^{n}f(x)dx}=m^{\alpha}.$$

We have the following inequalities for "large" integer $n > 1$ and integer $m > 1$

$$\frac{f(nm)}{f(n-1)}\leq\frac{p_{nm}}{p_n}\leq\frac{f(nm-1)}{f(n)}. \tag{4.5}$$

Since

$$\lim_{k\to+\infty}\frac{f(k+1)}{f(k)}=1,$$

therefore, from (4.5) we conclude

$$\lim_{n\to+\infty}\frac{P_{nm}}{P_n}=\lim_{n\to+\infty}\frac{f(nm)}{f(n)}=m^{\alpha}.$$

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Հաճախականային բաշխումները աճող կենսամոլեկուլար համակարգորում հիմնված կայուն խտությունների վրա

Մեծածավալ աճող կենսամոլեկուլյար համակարգերում շարունակվում են բաշխումների որոնումները՝ դրանցում առաջացող էմպիրիկ բաշխումները մոտարկելու համար: Ելնելով կենսամոլեկուլյար համակարգերի մասին մեծաքանակ տվյալներից՝ որոշ ընդհանուր վիճակագրական փաստեր են հաստատվել էմպիրիկ հաճախականային բաշխումների վերաբերյալ:

Այս աշխատանքում, որպես հաճախականային բաշխումների անընդհատ նմանակներ, առաջարկում ենք խտությունների երկպարամետրանի ընտանիքներ՝ կապված կայուն բաշխումների հետ: Ցույց ենք տալիս, որ այդ ընտանիքները բավարարում են կենսամոլեկուլյար համակարգերում առաջացող հաճախականային բաշխումների համար ստացված բոլոր վիճակագրական փաստերին:

Այդ ընտանիքների բաշխման ֆունկցիաների համար ստանում ենք ինտեգրալ ներկայացումներ, որոնք ավելի նպատակահարմար են այդ ընտանիքների հատկություններն ուսումնասիրելու և էմպիրիկ բաշխումները մոտարկելու համար: Հիմնվելով այս կայուն խտությունների վրա՝

դիսկրետիզացման միջոցով կառուցում ենք հաճախականային բաշխումներ, որոնք նույնպես բավարարում են հայտնի վիճակագրական փաստերին:

Äæ. Àñòîëà, Ý. À. Äàíèåëÿí, À. Ã. Àðàêåëÿí

×àñòîòíûå ðàñïðåäåëåíèÿ â ðàñòóùèõ áèîìîëåêóëÿðíûõ ñèñòåìàõ, îñíîâàííûå íà óñòîé÷èâûõ ïëîòíîñòÿõ

В растущих биомолекулярных системах больших размерностей продолжаются поиски распределений, которые могут служить аппроксимациями для возникающих в них эмпирических распределений. Исходя из огромных баз данных для таких систем установлены некоторые общие статистические факты эмпирических частотных распределений.

В данной работе в качестве непрерывных аналогов частотных распределений предложены двухпараметрические семейства плотностей, связанные с устойчивыми законами. Показано, что эти семейства удовлетворяют известным статистическим свойствам эмпирических распределений, возникающих в биомолекулярных системах.

Для функций распределений этих семейств получены интегральные представления, удобные при изучении их свойств и для апроксимации эмпирических распределеий. На основе устойчивых плотностей с помощью дискретизации строятся частотные распределения, которые также удовлетворяют известным статистическим свойствам.

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